Velocity Dependent Air Resistance

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1 Introduction

In this paper we will derive the equations of motion for a particle undergoing velocity dependent air resistance. For objects experiencing small amounts of air resistance, this is a decent approximation, but for objects experiencing large amounts of air resistance (due to high velocity, traveling through a fluid with a high viscosity, or for irregularly shaped objects for example), velocity squared dependent air resistance offers a better approximation. For a treatment of that case, the reader is encouraged to read [1], [2], and [3].

2 Derivation of the Equations of Motion

Consider a particle of mass m in free-fall with a speed $v(0) = v_0$ undergoing a force of air resistance proportional to it's velocity. By applying Newton's 2nd Law of motion to the particle, we can say that

$$ma = mg - bv \tag{1}$$

where b is an arbitrary coefficient of air resistance. Using the definition of acceleration, we can re-write (1) as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv \tag{2}$$

so that Newton's second law of motion for the particle is written in terms of the particle's velocity. By dividing each term in (2) by b, we get

$$\frac{m}{b}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{mg}{b} - v \tag{3}$$

In the next step, we will do two things: First, we will multiply both sides by $\mathrm{d}t$ to get $\mathrm{d}t$ on the right hand side. In addition, we will divide both sides by $\frac{mg}{b} - v$. This gives us

$$\frac{\mathrm{d}v}{(\frac{mg}{b} - v)} = \frac{b}{m}\mathrm{d}t \tag{4}$$

We then multiply both sides of (4) by -1 to turn equation (4) into the form of

$$\frac{\mathrm{d}v}{(v - \frac{mg}{b})} = -\frac{b}{m}\mathrm{d}t \tag{5}$$

Since m, g, b are constants, this is a separated differential equation where we can integrate both sides. By taking the integral of both sides,

$$\int \frac{\mathrm{d}v}{(v - \frac{mg}{b})} = \int -\frac{b}{m}\mathrm{d}t \tag{6}$$

We find that

$$\ln|v - \frac{mg}{b}| = -\frac{b}{m}t + C \tag{7}$$

Where C is a constant of integration. To solve for v, we then raise all terms by e, giving

$$v - \frac{mg}{b} = e^{-\frac{b}{m}t} + e^{C} \tag{8}$$

Which can be re-written using the definitions of exponents as

$$v - \frac{mg}{b} = e^{C}e^{-\frac{b}{m}t} \tag{9}$$

Which can be rearranged as

$$v(t) = e^{C}e^{-\frac{b}{m}t} + \frac{mg}{b} \tag{10}$$

We will now solve for the constant of integration by using the initial constraint that at $t = 0, v(0) = v_0$. Since C is an arbitrary constant of integration, e^C is also a constant, so we will just call $e^C = C$ as it is simply a constant of some value.

$$v(t) = Ce^{-\frac{b}{m}t} + \frac{mg}{b} \tag{11}$$

$$v_0 = Ce^{-\frac{b}{m}0} + \frac{mg}{b} \tag{12}$$

$$v_0 = C + \frac{mg}{b} \tag{13}$$

$$C = \frac{mg}{b} - v_0 \tag{14}$$

Therefore, we can express the equation for the velocity of a particle under velocity-dependent air resistance as

$$v(t) = (\frac{mg}{b} - v_0)e^{-\frac{b}{m}t} + \frac{mg}{b} \tag{15}$$

If we consider the case where $v_0 = 0$ and perform some simplifications, we arrive at

$$v(t) = \frac{mg}{b}(e^{-\frac{b}{m}t} + 1) \tag{16}$$

We will now consider the case where the particle is falling for a large period of time. As t gets large, we see that

$$\lim e^{-\frac{b}{m}t} \to 0 \tag{17}$$

Thus, for large amounts of time

$$v(t) \approx \frac{mg}{b} \tag{18}$$

which is a constant. Thus we have shown that for an object undergoing velocity dependent air resistance, the velocity approaches a constant velocity, which will call the **terminal velocity** and will denote as v_t

Thus $v_t = \frac{mg}{b}$ from which we can determine the drag coefficient of an object by solving for b which gives us

$$b = \frac{mg}{v_t} \tag{19}$$

3 Conclusion

In this paper we have used Newton's Laws of motion to derive an equation to give the velocity of an object undergoing velocity-dependent air resistance. We have shown that for large times (or large drag coefficients or small mass) an object in free fall which approach a constant terminal velocity and we determined it's value. We also demonstrated how to determine the drag coefficient for an object based upon it's mass and terminal velocity. For the interested reader, one can integrate the velocity equation and derive an equation for the position of an object in free fall under velocity dependent air resistance quite readily.

References

[1] $https://www.grc.nasa.gov/www/k-12/airplane/drageq.html$

[2] *https : //en.wikipedia.org/wiki/Drag(physics)*

[3] *http : //www.dehn.wustl.edu/ blake/courses/WU − 217 − 2010 − Fall/handouts/Rockets.pdf*