5 Integration and Its Applications

5.4 FURTHER APPLICATIONS OF DEFINITE INTEGRALS: AVERAGE VALUE AND AREA BETWEEN CURVES

Introduction

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- In this section we will use definite integrals for two important purposes: finding average values of functions and finding areas between curves.

- Average values are used everywhere. Birth weights of babies are compared with average weights, and retirement benefits are determined by average income.

- Averages eliminate fluctuations, reducing a collection of numbers to a single “representative” number.

- Areas between curves are used to find quantities from trade deficits to lives saved by seat belts.

Average Value of a Function

Average Value of a Function

The average value of *n* numbers is found by adding the numbers and dividing by *n*. For example,

$$\begin{pmatrix}\text{Average of} \\ a, b, \text{and } c\end{pmatrix} = \frac{1}{3}(a + b + c)$$

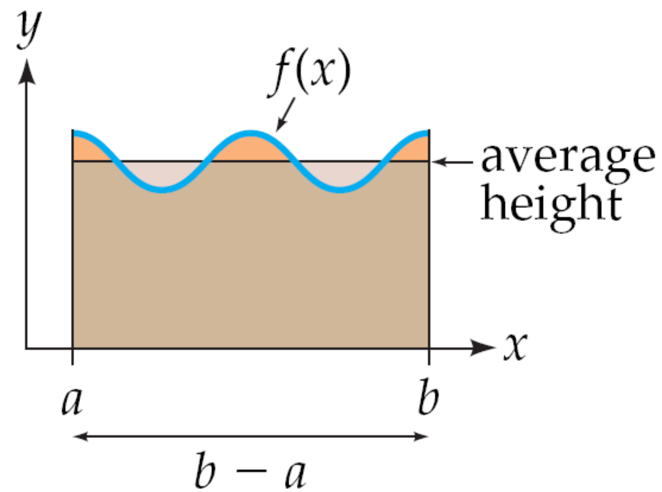
How can we find the average value of a *function* on an interval?

For example, if a function gives the temperature over a 24-hour period, how can we calculate the *average temperature?*

We could, of course, just take the temperature at every hour and then average these 24 values, but this would ignore the temperature at all of the intermediate times.

Average Value of a Function

Intuitively, the average should represent a "leveling off" of the curve to a uniform height, the horizontal line shown on the right.



This leveling should use the "hills" to fill in the "valleys," maintaining the same total area under the curve. Therefore, the area under the horizontal line must equal the area under the curve.

$$\underbrace{(b-a)\begin{pmatrix}\text{Average}\\ \text{height}\end{pmatrix}}_{\text{Area under line}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Area under curve}}$$

Equating the two areas

Average Value of a Function

Therefore:

$$\begin{pmatrix}\text{Average}\\ \text{height}\end{pmatrix} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Dividing by (b – a)

This formula gives the average (or "mean") value of a continuous function on an interval.

Average Value of a Function

$$\begin{pmatrix}\text{Average value}\\ \text{of } f \text{ on } [a, b]\end{pmatrix} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Average value is the definite integral of the function divided by the length of the interval

Finding the average value of a function by integrating and dividing by b – a is analogous to averaging n numbers by adding and dividing by n.

Example 1 – *FINDING THE AVERAGE VALUE OF A FUNCTION*

Find the average value of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 9$

Solution:

$$\begin{pmatrix}\text{Average}\\ \text{value}\end{pmatrix} = \frac{1}{9-0}\int_0^9 \sqrt{x}\,dx$$

Integral divided by the length of the interval

$$= \frac{1}{9}\int_0^9 x^{1/2}\,dx$$

$$= \frac{1}{9}\frac{2}{3}x^{3/2}\Big|_0^9$$

Integrating and evaluating

$$= \frac{2}{27}9^{3/2} - \frac{2}{27}0^{3/2}$$

Example 1 – *Solution*

cont'd

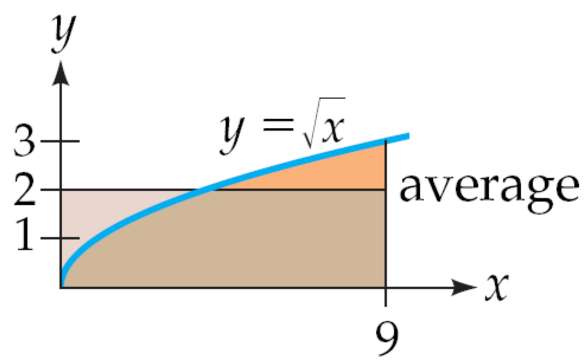
$$= \frac{2}{27}\left(\sqrt{9}\right)^3 - 0$$

Simplifying

$$= \frac{2}{27}27$$

$$= 2$$

The average value of $f(x) = \sqrt{x}$ over the interval [0, 9] is 2.

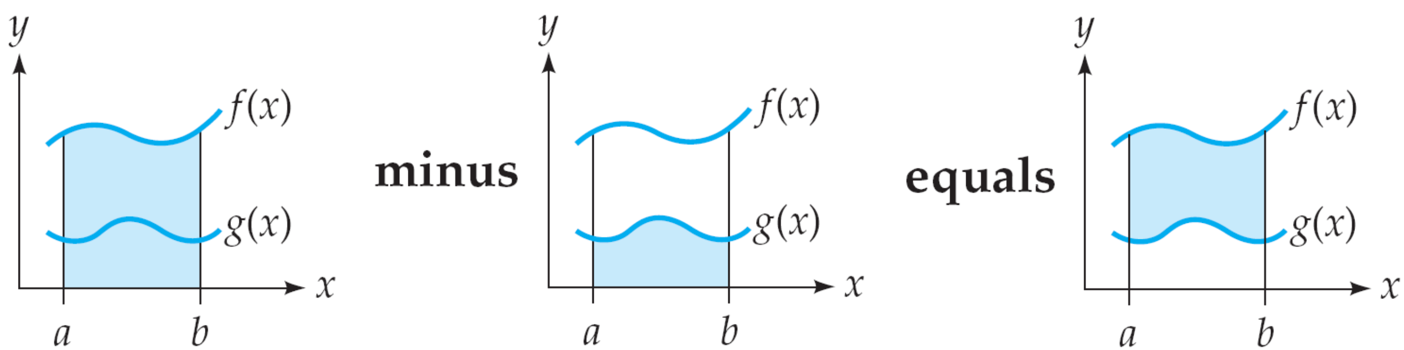


Area Between Curves: Integrating "Upper Minus Lower"

Area Between Curves: Integrating “Upper Minus Lower”

We know that definite integrals give areas under curves.

To calculate the area *between* two curves, we take the area under the *upper* curve and subtract the area under the *lower* curve.



In terms of integrals:

$$\int_a^b f(x)\,dx \qquad - \qquad \int_a^b g(x)\,dx \qquad = \qquad \int_a^b \Big[\underset{\substack{\uparrow \\ \text{Upper} \\ \text{curve}}}{f(x)} - \underset{\substack{\uparrow \\ \text{Lower} \\ \text{curve}}}{g(x)}\Big]\,dx$$

Area Between Curves: Integrating “Upper Minus Lower”

Therefore, the area between the curves can be written as a single integral:

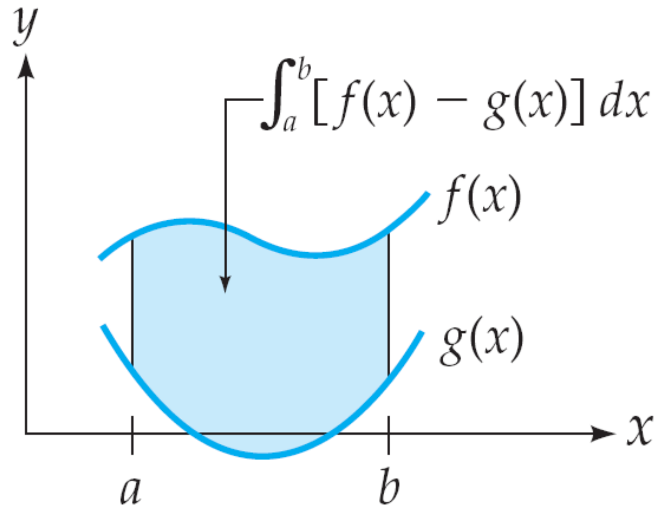
Area Between Curves

The area between two continuous curves $f(x) \geq g(x)$ on $[a, b]$ is

$$\begin{pmatrix} \text{Area between} \\ f \text{ and } g \text{ on } [a, b] \end{pmatrix} = \int_a^b [f(x) - g(x)]\, dx$$

Integrate “upper minus lower”

Finding area by integrating “upper minus lower” works regardless of whether one or both curves dip below the *x*- axis.

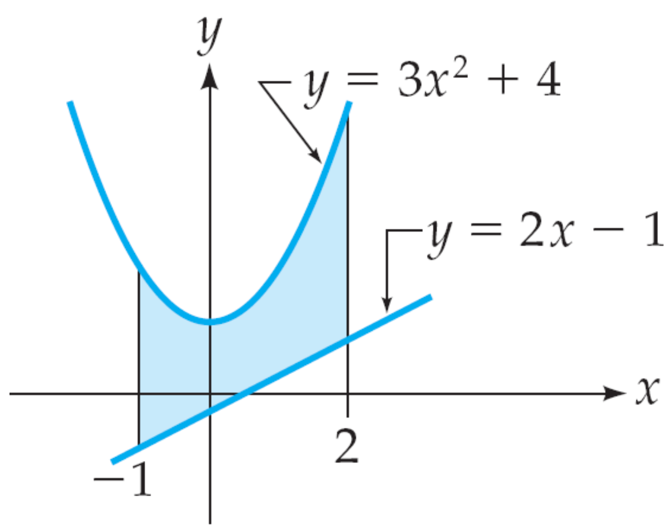


Example 3 – *FINDING THE AREA BETWEEN CURVES*

Find the area between $y = 3x^2 + 4$ and $y = 2x – 1$ from $x = –1$ to $x = 2$.

Solution:

The area is shown in the diagram on the left. We integrate "upper minus lower" between the given x-values.



Example 1 – *Solution*

cont'd

$$\int_{-1}^{2} [\underbrace{(3x^2 + 4)}_{\text{Upper}} - \underbrace{(2x - 1)}_{\text{Lower}}]\, dx$$

$$= \int_{-1}^{2} (3x^2 + 4 - 2x + 1)\, dx$$

$$= \int_{-1}^{2} (3x^2 - 2x + 5)\, dx$$

Simplifying

$$= (x^3 - x^2 + 5x)\Big|_{-1}^{2}$$

Integrating

Example 1 – *Solution*

cont'd

$$= (8 - 4 + 10) - (-1 - 1 - 5)$$ Evaluating

= 21 square units

Area Between Curves: Integrating “Upper Minus Lower”

If the two curves represent *rates* (one unit per another unit), then the area between the curves gives the *total accumulation* at the upper rate minus the lower rate.

Area Between Curves That Cross

Area Between Curves That Cross

At a point where two curves cross, the upper curve becomes the lower curve and the lower becomes the upper.

The area between them must then be calculated by two (or more) integrals, upper minus lower on *each* interval.

Example 5 – *FINDING THE AREA BETWEEN CURVES THAT CROSS*

Find the area between the curves $y = 12 - 3x^2$ and $y = 4x + 5$ from x = 0 to $x = 3$.

Solution:

A sketch shows that the curves *do* cross. To find the intersection point, we set the functions equal to each other and solve.

$$4x + 5 = 12 - 3x^2$$ Equating the two functions

$$3x^2 + 4x - 7 = 0$$ Combining all terms on the left

$$(3x + 7)(x - 1) = 0$$ Factoring

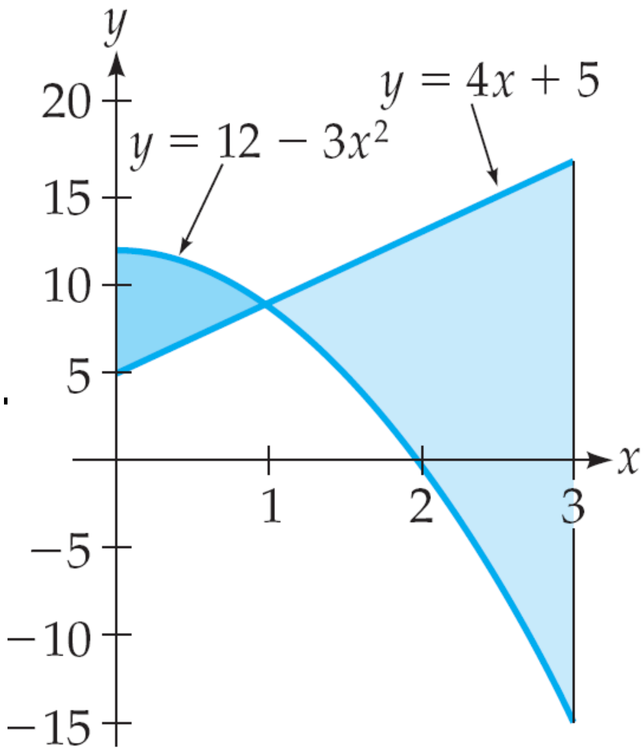
Example 5 – *Solution*

cont'd

$$x = 1$$

The other solution, $x = -7/3$, is not in the interval [0, 3]

The curves cross at $x = 1$, so we must integrate separately over the intervals [0, 1] and [1, 3]. The diagram shows which curve is upper and which is lower on each interval.



Example 5 – *Solution*

cont'd

$$\int_0^1 [\underbrace{(12 - 3x^2)}_{\text{Upper}} - \underbrace{(4x + 5)}_{\text{Lower}}]\,dx + \int_1^3 [\underbrace{(4x + 5)}_{\text{Upper}} - \underbrace{(12 - 3x^2)}_{\text{Lower}}]\,dx$$

Integrating upper minus lower on each interval

$$= \int_0^1 (-3x^2 - 4x + 7)\,dx + \int_1^3 (3x^2 + 4x - 7)\,dx$$

Simplifying the integrands

$$= (-x^3 - 2x^2 + 7x)\Big|_0^1 + (x^3 + 2x^2 - 7x)\Big|_1^3$$

Integrating and simplifying

$$= \overbrace{-1 - 2 + 7}^{} + \overbrace{27 + 18 - 21 - (1 + 2 - 7)}^{} = 32$$

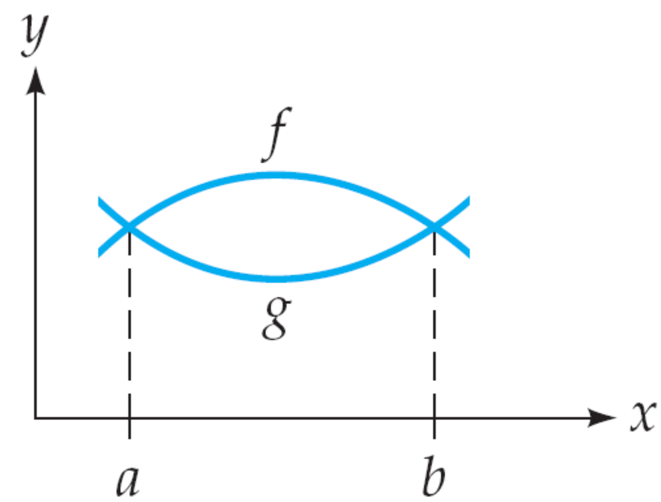
Evaluating

Therefore, the area between the curves is 32 square units.

Areas Bounded by Curves

Areas Bounded by Curves

It is sometimes useful to find the *area bounded by two curves,* without being told the starting and ending x-values. In such problems the curves completely enclose an area, and the x-values for the upper and lower limits of integration are found by setting the functions equal to each other and solving.



The x-values a and b are where the curves meet.

Example 6 – *FINDING AN AREA BOUNDED BY CURVES*

Find the area bounded by the curves

$$y = 3x^2 - 12 \text{ and } y = 12 - 3x^2$$

Solution:

The x-values for the upper and lower limits of integration are not given, so we find them by setting the functions equal to each other and solving.

$3x^2 - 12 = 12 - 3x^2$	Setting the functions equal to each other
$6x^2 - 24 = 0$	Combining everything on one side
$6(x^2 - 4) = 0$	Factoring
$6(x + 2)(x - 2) = 0$	Factoring further
$x = -2$ and $x = 2$	Solving

Example 6 – *Solution*

cont'd

The smaller of these, $x = -2$, is the lower limit of integration and the larger, $x = 2$, is the upper limit. To determine which function is "upper" and which is "lower," we choose a "test value" between $x = -2$ and $x = 2$ ($x = 0$ will do), which we substitute into each function to see which is larger.

Evaluating each of the original functions at the test point $x = 0$ yields

$3x^2 - 12 = 3(0)^2 - 12 = -12$ (Smaller) $3x^2 - 12$ at $x = 0$

$12 - 3x^2 = 12 - 3(0)^2 = 12$ (Larger) $12 - 3x^2$ at $x = 0$

Example 6 – *Solution*

cont'd

Therefore, $y = 12 - 3x^2$ is the "upper" function (since it gives a higher y-value) and $y = 3x^2 - 12$ is the "lower" function. We then integrate upper minus lower between the x-values found earlier.

$$\int_{-2}^{2} [\underbrace{12 - 3x^2}_{\text{Upper}} - (\underbrace{3x^2 - 12}_{\text{Lower}})]\, dx$$

Simplifying

$$= \int_{-2}^{2} (24 - 6x^2)\, dx$$

$$= (24x - 2x^3) \Big|_{-2}^{2}$$

Integrating

Example 6 – *Solution*

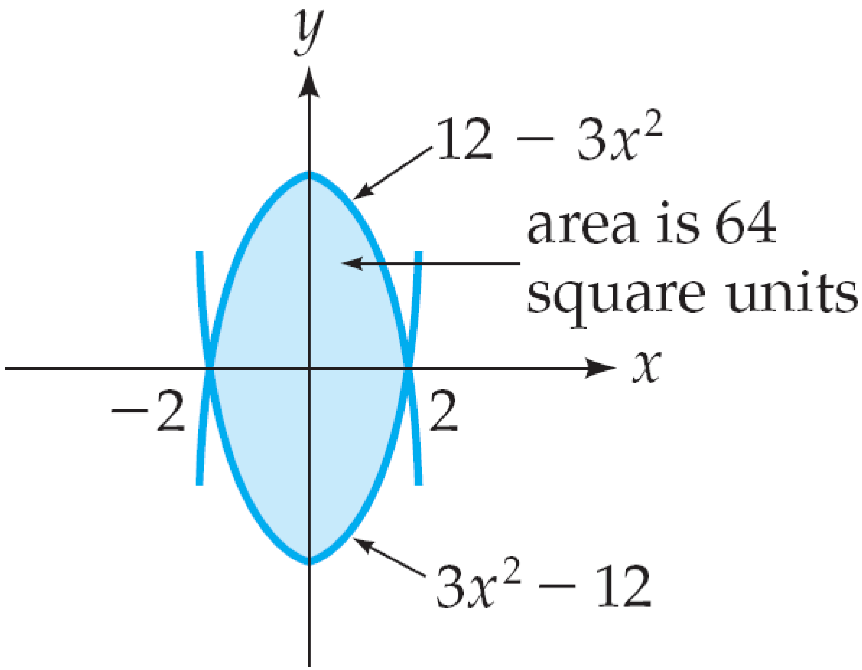
cont'd

$$= (48 - 16) - (-48 + 16) = 64$$ Evaluating

The area bounded by the two curves is 64 square units.

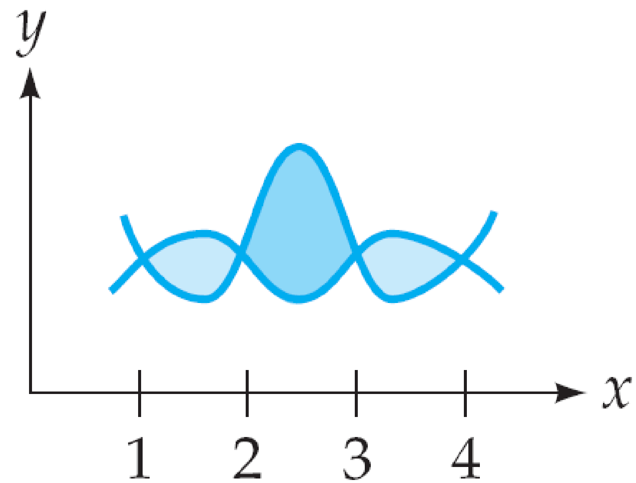
Areas Bounded by Curves

The two curves $y = 12 - 3x^2$ and $y = 3x^2 - 12$ and are shown in the graph below. Notice that we were able to calculate the area between them without having to graph them.



Areas Bounded by Curves

For curves that intersect at *more* than two points, several integrals may be needed, integrating “upper minus lower” on each interval, as in Example 5. Test points in each interval will determine the upper and lower curves on that interval.



Integrate *upper minus lower* on three intervals.