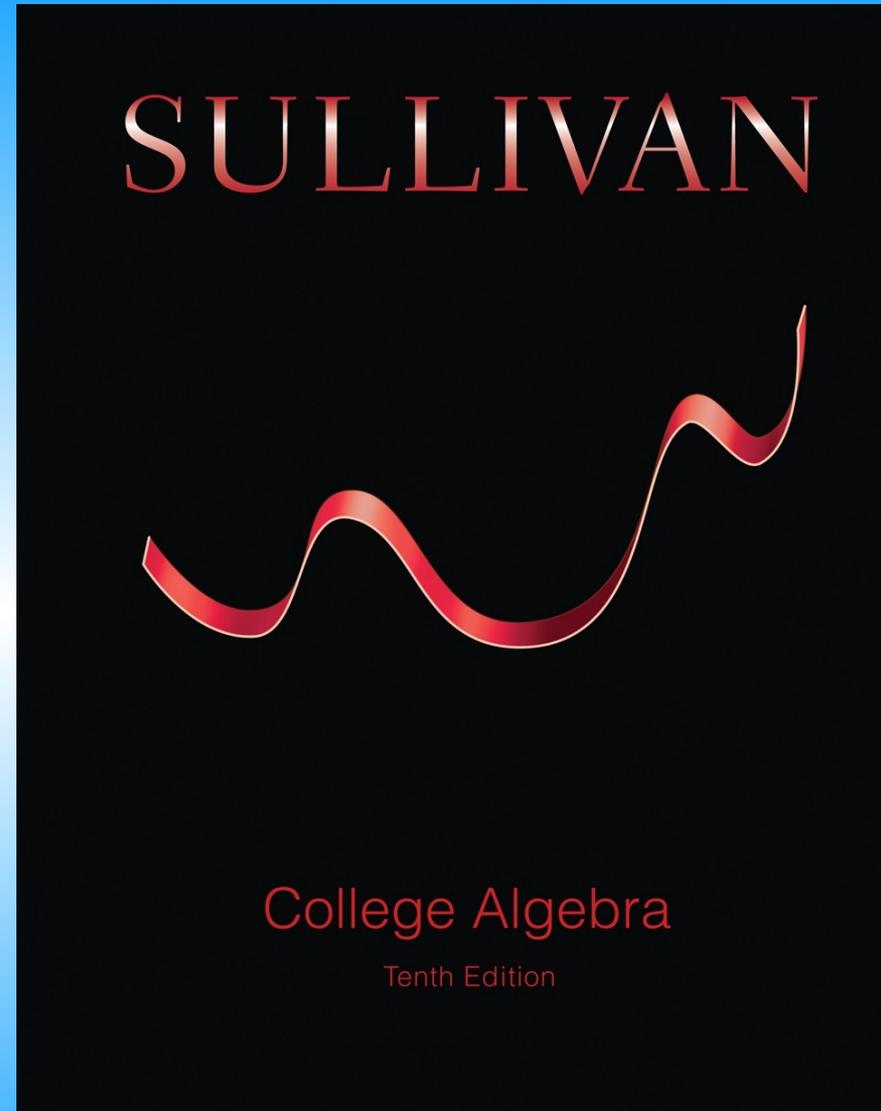


# Chapter 3

## Section 3



## 3.3 Properties of Functions

**PREPARING FOR THIS SECTION** Before getting started, review the following:

- Intervals (Section 1.5, pp. 120–121)
- Intercepts (Section 2.2, pp. 159–160)
- Slope of a Line (Section 2.3, pp. 167–169)
- Point–Slope Form of a Line (Section 2.3, p. 171)
- Symmetry (Section 2.2, pp. 160–162)



**Now Work** the 'Are You Prepared?' problems on page 232.

- OBJECTIVES**
- 1** Determine Even and Odd Functions from a Graph (p. 223)
  - 2** Identify Even and Odd Functions from an Equation (p. 225)
  - 3** Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant (p. 225)
  - 4** Use a Graph to Locate Local Maxima and Local Minima (p. 226)
  - 5** Use a Graph to Locate the Absolute Maximum and the Absolute Minimum (p. 227)
  -  **6** Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing (p. 229)
  - 7** Find the Average Rate of Change of a Function (p. 230)

# Determine Even and Odd Functions from a Graph

# Definition

---

A function  $f$  is **even** if, for every number  $x$  in its domain, the number  $-x$  is also in the domain and

$$f(-x) = f(x)$$

# Definition

---

A function  $f$  is **odd** if, for every number  $x$  in its domain, the number  $-x$  is also in the domain and

$$f(-x) = -f(x)$$

# Theorem

---

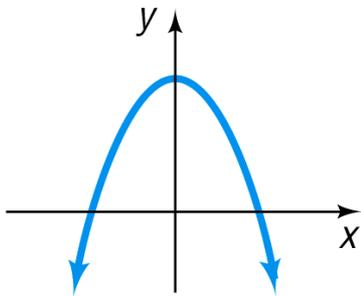
A function is even if and only if its graph is symmetric with respect to the  $y$ -axis. A function is odd if and only if its graph is symmetric with respect to the origin.

# Example

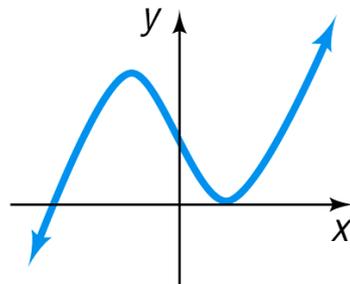
---

## Determining Even and Odd Functions from the Graph

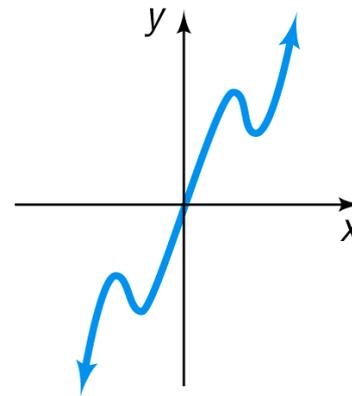
Determine whether each graph given in Figure 17 is the graph of an even function, an odd function, or a function that is neither even nor odd.



(a)



(b)



(c)

Figure 17

# Solution

---

- (a) The graph in Figure 17(a) is that of an even function, because the graph is symmetric with respect to the  $y$ -axis.
- (b) The function whose graph is given in Figure 17(b) is neither even nor odd, because the graph is neither symmetric with respect to the  $y$ -axis nor symmetric with respect to the origin.
- (c) The function whose graph is given in Figure 17(c) is odd, because its graph is symmetric with respect to the origin.

# Identify Even and Odd Functions from an Equation

# Example

---

## Identifying Even and Odd Functions Algebraically

Determine whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the  $y$ -axis, with respect to the origin, or neither.

(a)  $f(x) = x^2 - 5$

(b)  $g(x) = x^3 - 1$

(c)  $h(x) = 5x^3 - x$

(d)  $F(x) = |x|$

# Solution

---

- (a) To determine whether  $f$  is even, odd, or neither, replace  $x$  by  $-x$  in  $f(x) = x^2 - 5$ .

$$f(-x) = (-x)^2 - 5 = x^2 - 5 = f(x)$$

Since  $f(-x) = f(x)$ , the function is even, and the graph of  $f$  is symmetric with respect to the  $y$ -axis.

- (b) Replace  $x$  by  $-x$  in  $g(x) = x^3 - 1$ .

$$g(-x) = (-x)^3 - 1 = -x^3 - 1$$

Since  $g(-x) \neq g(x)$  and  $g(-x) \neq -g(x) = -(x^3 - 1) = -x^3 + 1$ , the function is neither even nor odd. The graph of  $g$  is not symmetric with respect to the  $y$ -axis, nor is it symmetric with respect to the origin.

# Solution continued

---

(c) Replace  $x$  by  $-x$  in  $h(x) = 5x^3 - x$ .

$$h(-x) = 5(-x)^3 - (-x) = -5x^3 + x = -(5x^3 - x) = -h(x)$$

Since  $h(-x) = -h(x)$ ,  $h$  is an odd function, and the graph of  $h$  is symmetric with respect to the origin.

(d) Replace  $x$  by  $-x$  in  $F(x) = |x|$ .

$$F(-x) = |-x| = |-1| \cdot |x| = |x| = F(x)$$

Since  $F(-x) = F(x)$ ,  $F$  is an even function, and the graph of  $F$  is symmetric with respect to the  $y$ -axis.

**Use a Graph to  
Determine Where a  
Function Is Increasing,  
Decreasing, or  
Constant**

# Example

## Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Determine the values of  $x$  for which the function in Figure 18 is increasing. Where is it decreasing? Where is it constant?

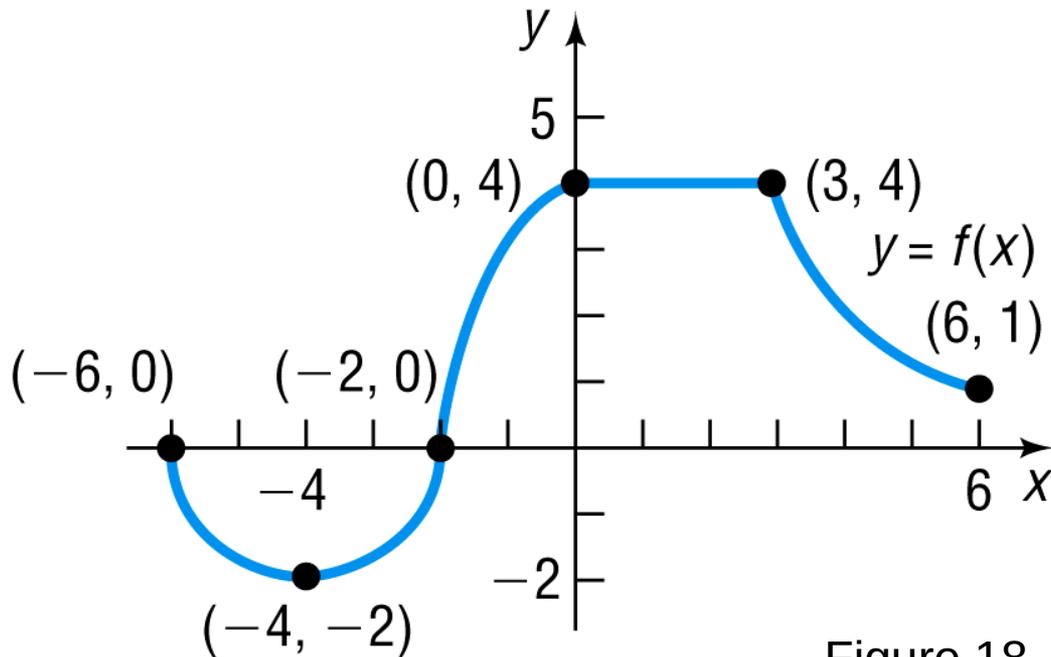


Figure 18

# Solution

---

When determining where a function is increasing, where it is decreasing, and where it is constant, we use strict inequalities involving the independent variable  $x$ , or we use open intervals\* of  $x$ -coordinates. The function whose graph is given in Figure 18 is increasing on the open interval  $(-4, 0)$ , or for  $-4 < x < 0$ . The function is decreasing on the open intervals  $(-6, -4)$  and  $(3, 6)$ , or for  $-6 < x < -4$  and  $3 < x < 6$ . The function is constant on the open interval  $(0, 3)$ , or for  $0 < x < 3$ .

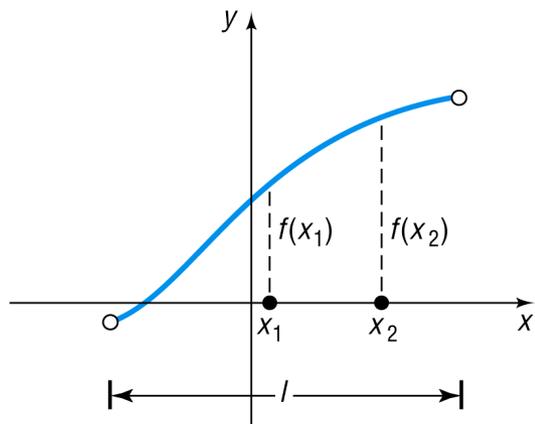
---

**DEFINITIONS** A function  $f$  is **increasing** on an open interval  $I$  if, for any choice of  $x_1$  and  $x_2$  in  $I$ , with  $x_1 < x_2$ , we have  $f(x_1) < f(x_2)$ .

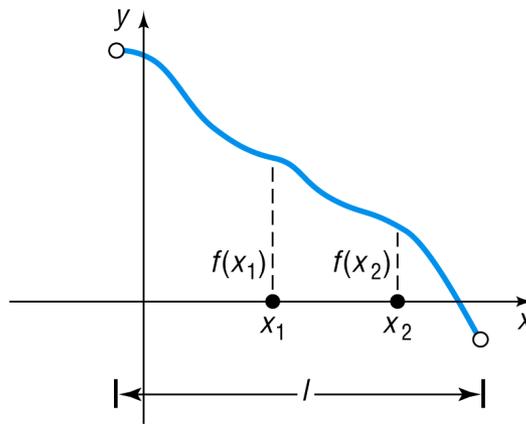
A function  $f$  is **decreasing** on an open interval  $I$  if, for any choice of  $x_1$  and  $x_2$  in  $I$ , with  $x_1 < x_2$ , we have  $f(x_1) > f(x_2)$ .

A function  $f$  is **constant** on an open interval  $I$  if, for all choices of  $x$  in  $I$ , the values  $f(x)$  are equal.

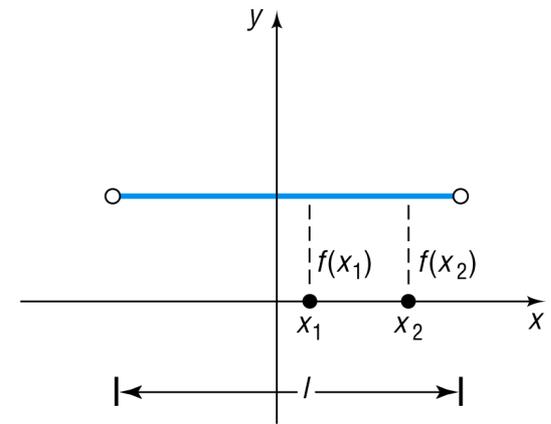
# Figure: Illustration of the Definitions



**(a)** For  $x_1 < x_2$  in  $I$ ,  
 $f(x_1) < f(x_2)$ ;  
 $f$  is increasing on  $I$ .



**(b)** For  $x_1 < x_2$  in  $I$ ,  
 $f(x_1) > f(x_2)$ ;  
 $f$  is decreasing on  $I$ .

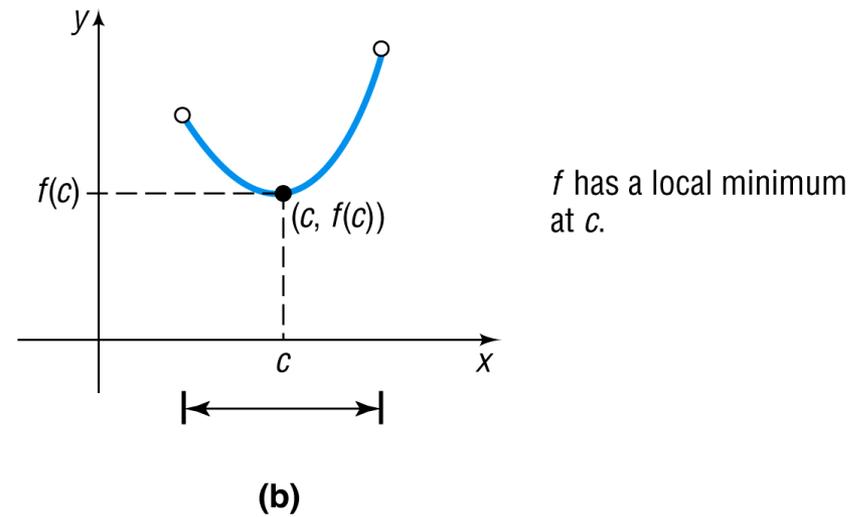
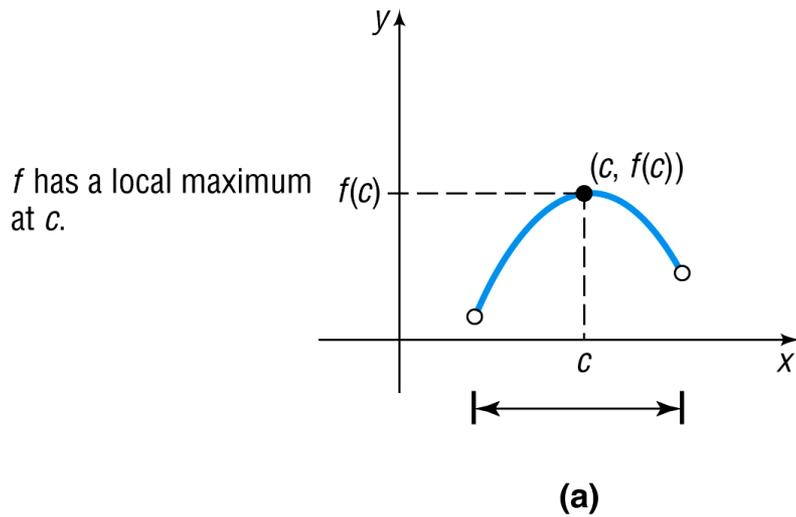


**(c)** For all  $x$  in  $I$ , the values of  
 $f$  are equal;  $f$  is constant on  $I$ .

# Use a Graph to Locate Local Maxima and Local Minima

# Figure: Local maximum and local minimum

---



# Definition

---

Let  $f$  be a function defined on some interval  $I$ .

A function  $f$  has a **local maximum** at  $c$  if there is an open interval in  $I$  containing  $c$  so that, for all  $x$  in this open interval, we have  $f(x) \leq f(c)$ . We call  $f(c)$  a **local maximum value of  $f$** .

A function  $f$  has a **local minimum** at  $c$  if there is an open interval in  $I$  containing  $c$  so that, for all  $x$  in this open interval, we have  $f(x) \geq f(c)$ . We call  $f(c)$  a **local minimum value of  $f$** .

# Example

---

## **Finding Local Maxima and Local Minima from the Graph of a Function and Determining Where the Function Is Increasing, Decreasing, or Constant**

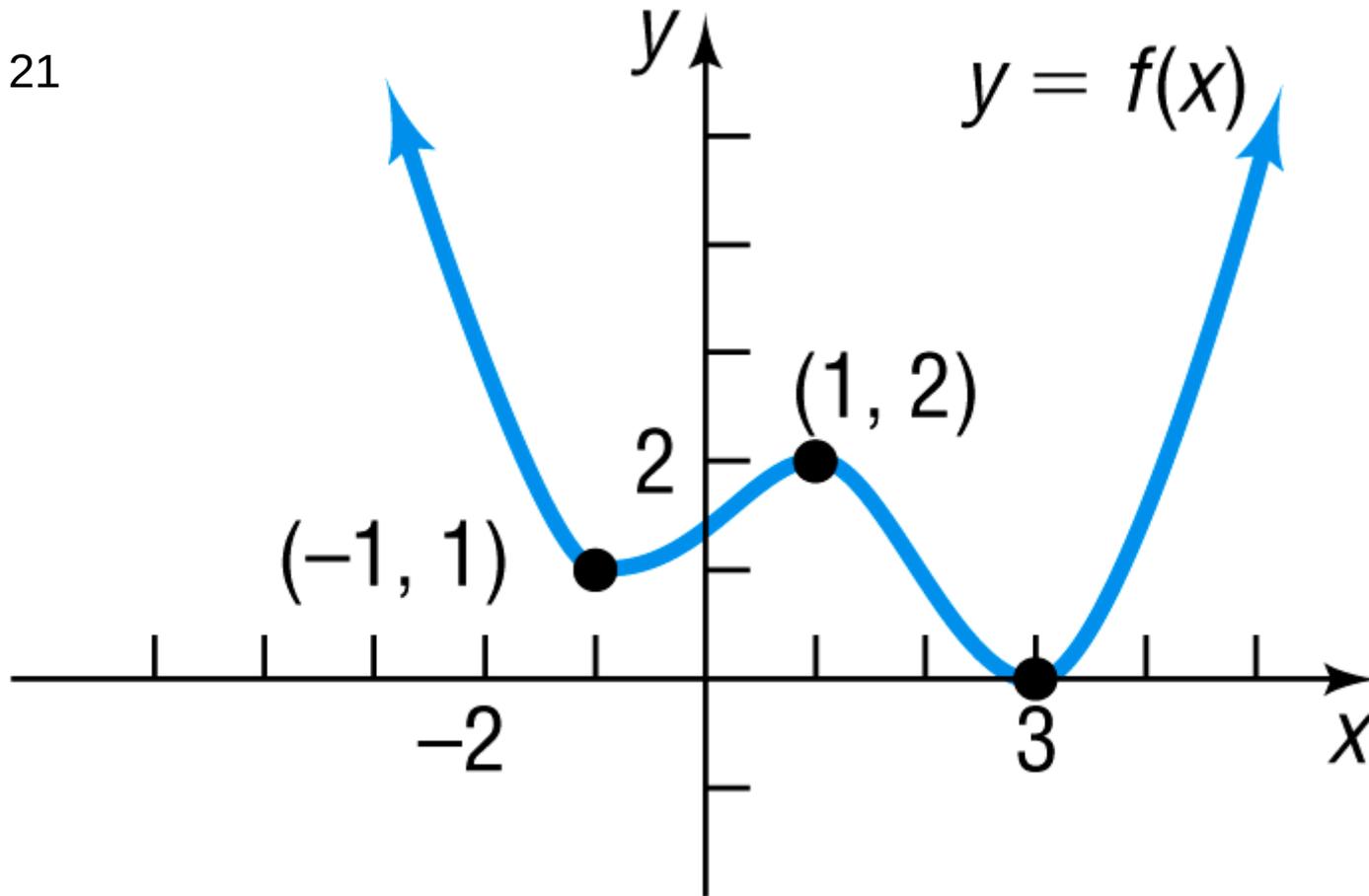
Figure 21 shows the graph of a function  $f$ .

- (a) At what value(s) of  $x$ , if any, does  $f$  have a local maximum? List the local maximum values.
- (b) At what value(s) of  $x$ , if any, does  $f$  have a local minimum? List the local minimum values.
- (c) Find the intervals on which  $f$  is increasing. Find the intervals on which  $f$  is decreasing.

# Example continued

---

Figure 21



# Solution

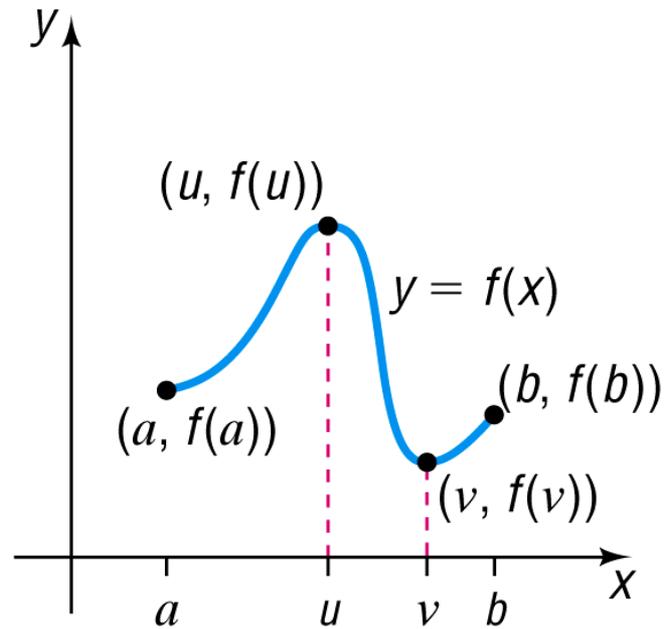
---

The domain of  $f$  is the set of real numbers.

- (a)  $f$  has a local maximum at 1, since for all  $x$  close to 1, we have  $f(x) \leq f(1)$ . The local maximum value is  $f(1) = 2$ .
- (b)  $f$  has local minima at  $-1$  and at 3. The local minimum values are  $f(-1) = 1$  and  $f(3) = 0$ .
- (c) The function whose graph is given in Figure 21 is increasing for all values of  $x$  between  $-1$  and 1 and for all values of  $x$  greater than 3. That is, the function is increasing on the intervals  $(-1, 1)$  and  $(3, \infty)$ , or for  $-1 < x < 1$  and  $x > 3$ . The function is decreasing for all values of  $x$  less than  $-1$  and for all values of  $x$  between 1 and 3. That is, the function is decreasing on the intervals  $(-\infty, -1)$  and  $(1, 3)$ , or for  $x < -1$  and  $1 < x < 3$ .

# Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

# Figure



domain:  $[a, b]$

for all  $x$  in  $[a, b]$ ,  $f(x) \leq f(u)$

for all  $x$  in  $[a, b]$ ,  $f(x) \geq f(v)$

absolute maximum:  $f(u)$

absolute minimum:  $f(v)$

---

**DEFINITION** Let  $f$  be a function defined on some interval  $I$ . If there is a number  $u$  in  $I$  for which  $f(x) \leq f(u)$  for all  $x$  in  $I$ , then  $f$  has an **absolute maximum at  $u$** , and the number  $f(u)$  is the **absolute maximum of  $f$  on  $I$** .

If there is a number  $v$  in  $I$  for which  $f(x) \geq f(v)$  for all  $x$  in  $I$ , then  $f$  has an **absolute minimum at  $v$** , and the number  $f(v)$  is the **absolute minimum of  $f$  on  $I$** .

# Example

## Finding the Absolute Maximum and the Absolute Minimum from the Graph of a Function

For each graph of a function  $y = f(x)$  in Figure 23, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.

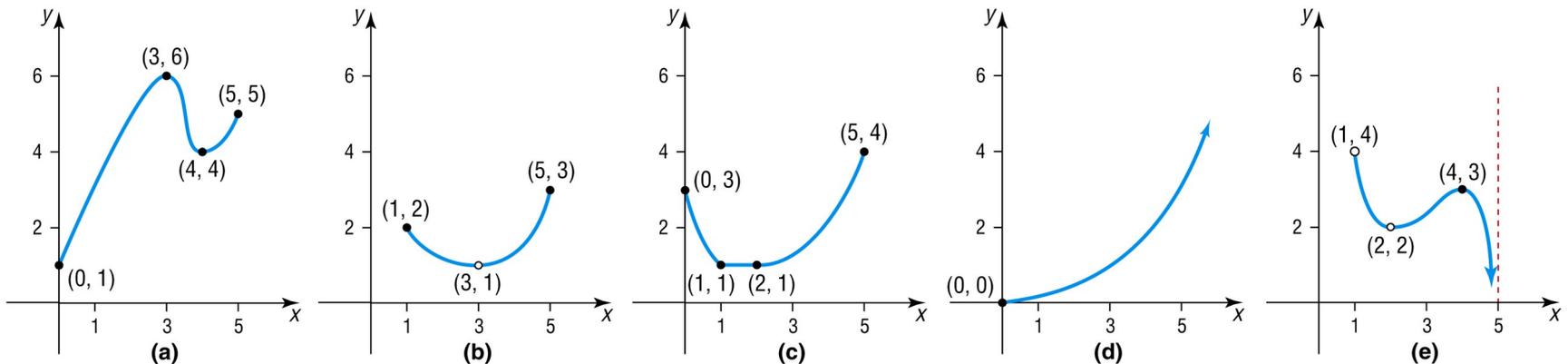


Figure 23

# Solution

---

- (a) The function  $f$  whose graph is given in Figure 23(a) has the closed interval  $[0, 5]$  as its domain. The largest value of  $f$  is  $f(3) = 6$ , the absolute maximum. The smallest value of  $f$  is  $f(0) = 1$ , the absolute minimum. The function has a local maximum of 6 at  $x = 3$  and a local minimum of 4 at  $x = 4$ .
- (b) The function  $f$  whose graph is given in Figure 23(b) has the domain  $\{x \mid 1 \leq x \leq 5, x \neq 3\}$ . Note that we exclude 3 from the domain because of the “hole” at  $(3, 1)$ . The largest value of  $f$  on its domain is  $f(5) = 3$ , the absolute maximum. There is no absolute minimum. Do you see why? As you trace the graph, getting closer to the point  $(3, 1)$ , there is no single smallest value. [As soon as you claim a smallest value, we can trace closer to  $(3, 1)$  and get a smaller value!] The function has no local maxima or minima.

# Solution continued

---

- (c) The function  $f$  whose graph is given in Figure 23(c) has the interval  $[0, 5]$  as its domain. The absolute maximum of  $f$  is  $f(5) = 4$ . The absolute minimum is 1. Notice that the absolute minimum 1 occurs at any number in the interval  $[1, 2]$ . The function has a local minimum value of 1 at every  $x$  in the interval  $[1, 2]$ , but it has no local maximum value.
- (d) The function  $f$  given in Figure 23(d) has the interval  $[0, \infty)$  as its domain. The function has no absolute maximum; the absolute minimum is  $f(0) = 0$ . The function has no local maximum or local minimum.
- (e) The function  $f$  in Figure 23(e) has the domain  $\{x \mid 1 < x < 5, x \neq 2\}$ . The function has no absolute maximum and no absolute minimum. Do you see why? The function has a local maximum value of 3 at  $x = 4$ , but no local minimum value.

# Theorem

---

## Extreme Value Theorem

If  $f$  is a continuous function\* whose domain is a closed interval  $[a, b]$ , then  $f$  has an absolute maximum and an absolute minimum on  $[a, b]$ .

**Use a Graphing Utility to  
Approximate Local  
Maxima and Local Minima  
and to Determine Where a  
Function Is Increasing or  
Decreasing**

# Example

---

## Using a Graphing Utility to Approximate Local Maxima and Minima and to Determine Where a Function Is Increasing or Decreasing

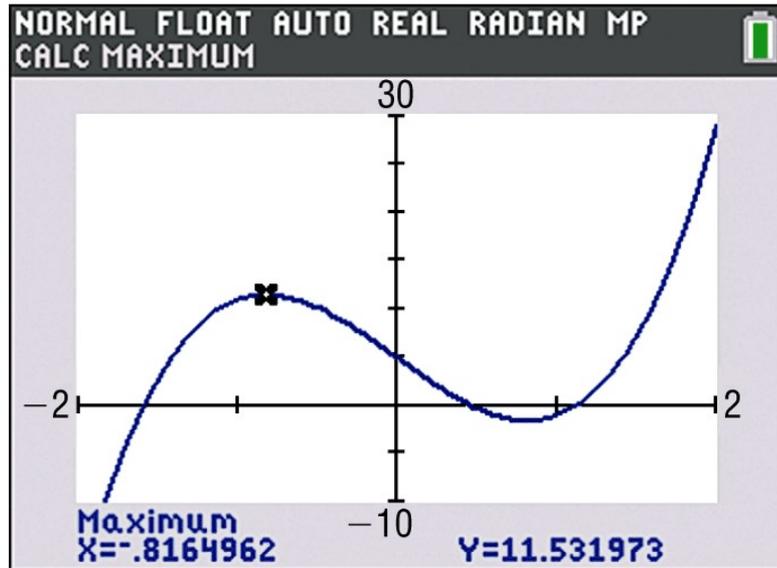
- (a) Use a graphing utility to graph  $f(x) = 6x^3 - 12x + 5$  for  $-2 < x < 2$ . Approximate where  $f$  has a local maximum and where  $f$  has a local minimum.
- (b) Determine where  $f$  is increasing and where it is decreasing.

# Solution

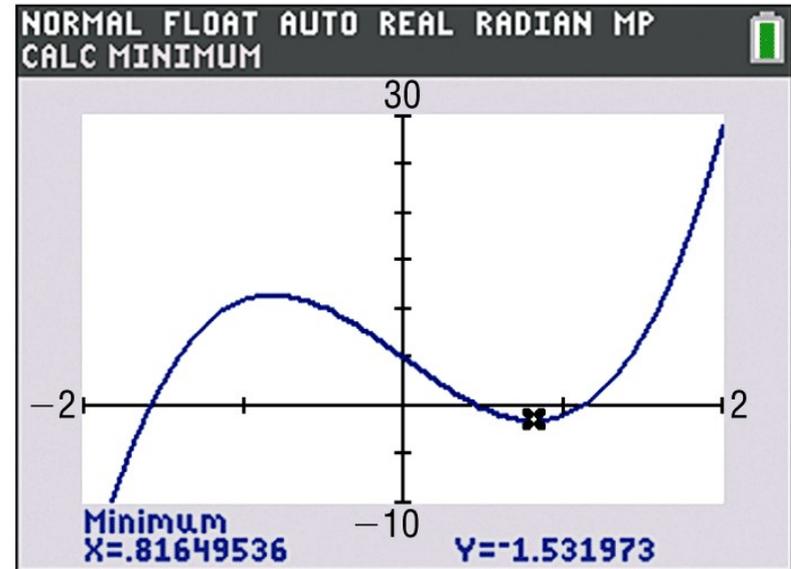
---

- (a) Graphing utilities have a feature that finds the maximum or minimum point of a graph within a given interval. Graph the function  $f$  for  $-2 < x < 2$ . The MAXIMUM and MINIMUM commands require us to first determine the open interval  $I$ . The graphing utility will then approximate the maximum or minimum value in the interval. Using MAXIMUM, we find that the local maximum value is 11.53 and that it occurs at  $x = -0.82$ , rounded to two decimal places. See Figure 24(a). Using MINIMUM, we find that the local minimum value is  $-1.53$  and that it occurs at  $x = 0.82$ , rounded to two decimal places. See Figure 24(b).
- (b) Looking at Figures 24(a) and (b), we see that the graph of  $f$  is increasing from  $x = -2$  to  $x = -0.82$  and from  $x = 0.82$  to  $x = 2$ , so  $f$  is increasing on the intervals  $(-2, -0.82)$  and  $(0.82, 2)$ , or for  $-2 < x < -0.82$  and  $0.82 < x < 2$ . The graph is decreasing from  $x = -0.82$  to  $x = 0.82$ , so  $f$  is decreasing on the interval  $(-0.82, 0.82)$ , or for  $-0.82 < x < 0.82$ .

# Solution continued



(a) Local maximum



(b) Local minimum

Figure 24

# Find the Average Rate of Change of a Function

# Definition

---

If  $a$  and  $b$ ,  $a \neq b$ , are in the domain of a function  $y = f(x)$ , the **average rate of change of  $f$**  from  $a$  to  $b$  is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \quad a \neq b \quad (1)$$

# Example

---

## Finding the Average Rate of Change

Find the average rate of change of  $f(x) = 3x^2$ :

(a) From 1 to 3

(b) From 1 to 5

(c) From 1 to 7

# Solution

---

(a) The average rate of change of  $f(x) = 3x^2$  from 1 to 3 is

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 3}{3 - 1} = \frac{24}{2} = 12$$

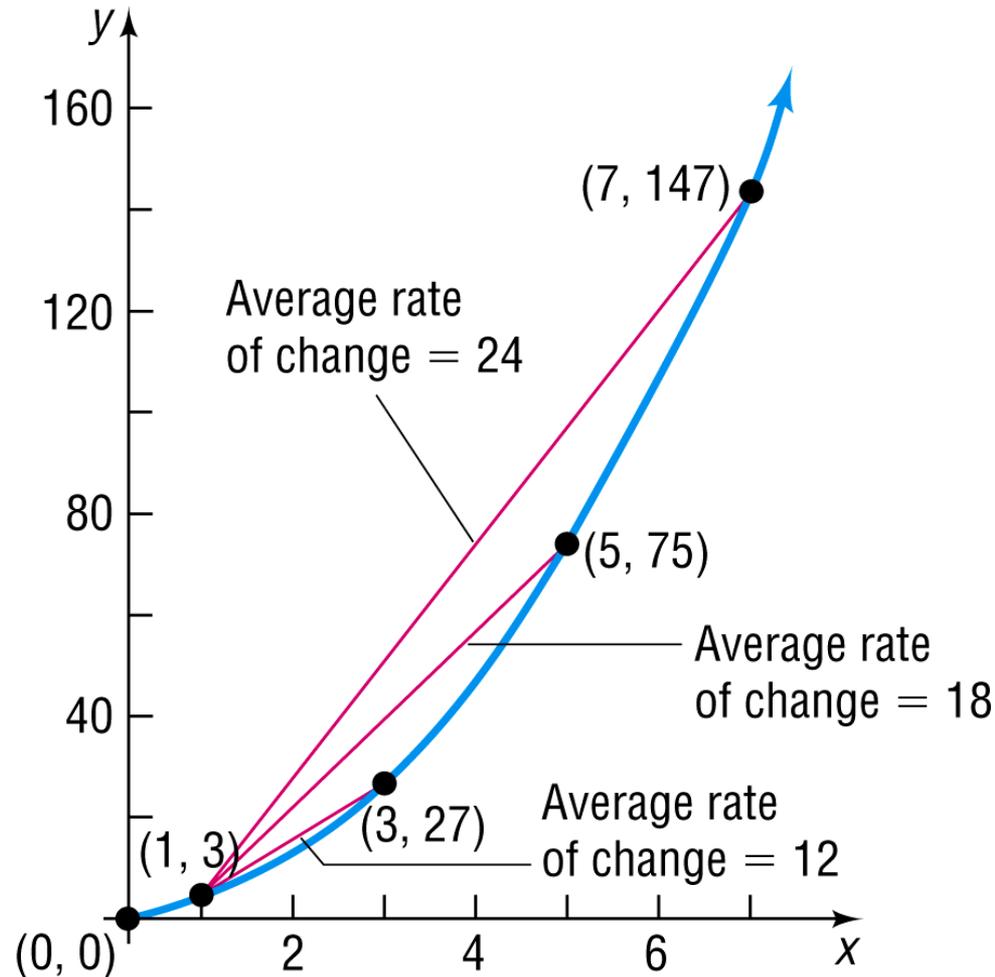
(b) The average rate of change of  $f(x) = 3x^2$  from 1 to 5 is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(1)}{5 - 1} = \frac{75 - 3}{5 - 1} = \frac{72}{4} = 18$$

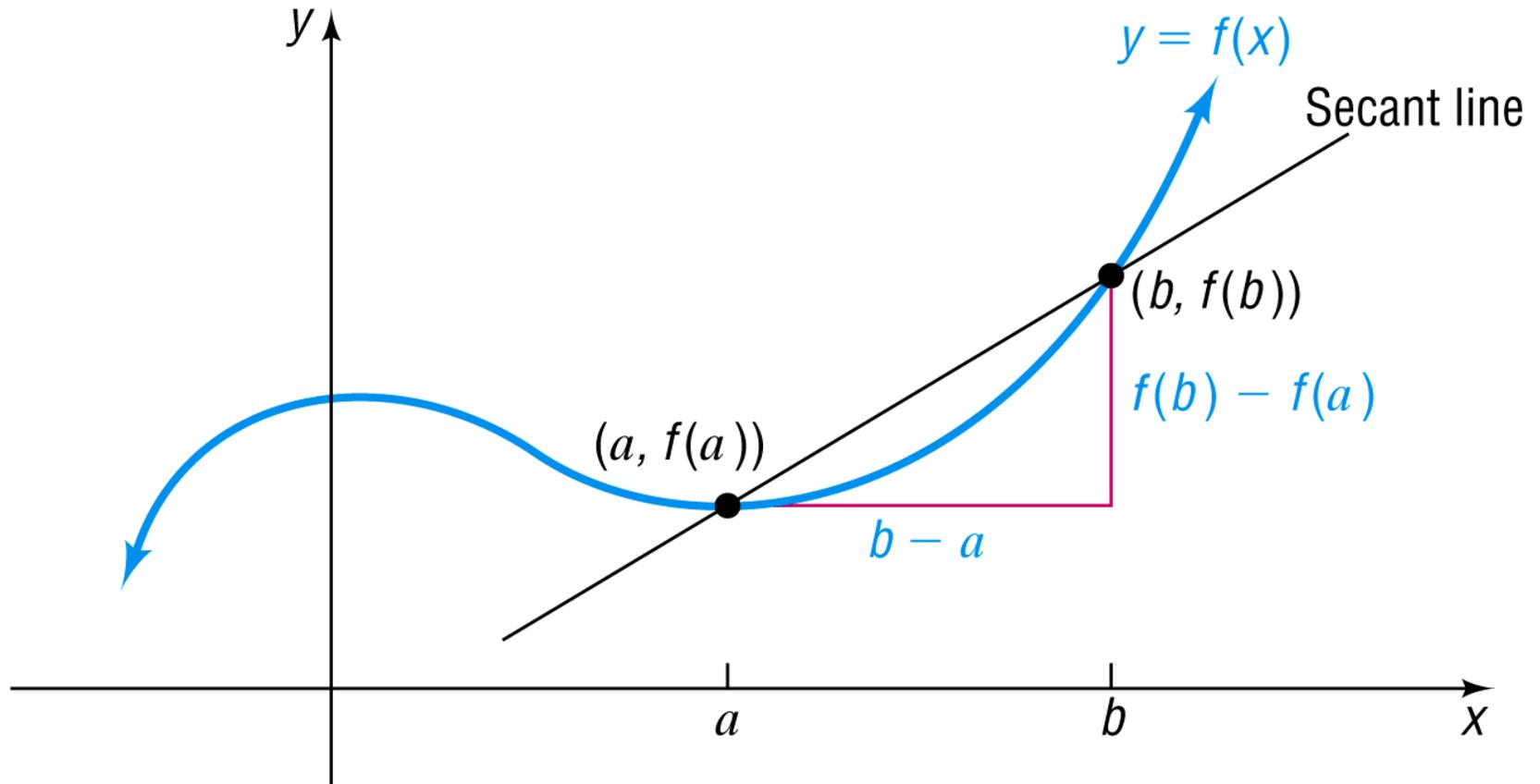
(c) The average rate of change of  $f(x) = 3x^2$  from 1 to 7 is

$$\frac{\Delta y}{\Delta x} = \frac{f(7) - f(1)}{7 - 1} = \frac{147 - 3}{7 - 1} = \frac{144}{6} = 24$$

# Figure: $f(x) = 3x^2$



# Figure: Secant Line



# Theorem

---

## Slope of the Secant Line

The average rate of change of a function from  $a$  to  $b$  equals the slope of the secant line containing the two points  $(a, f(a))$  and  $(b, f(b))$  on its graph.

# Example

---

## Finding the Equation of a Secant Line

Suppose that  $g(x) = 3x^2 + 2x - 1$ .

- (a) Find the average rate of change of  $g$  from  $-2$  to  $1$ .
- (b) Find an equation of the secant line containing  $(-2, g(-2))$  and  $(1, g(1))$ .
- (c) Draw the graph of  $g$  and the secant line obtained in part (b) on the same axes.

# Solution

---

(a) The average rate of change of  $g(x) = 3x^2 + 2x - 1$  from  $-2$  to  $1$  is

$$\begin{aligned}\text{Average rate of change} &= \frac{g(1) - g(-2)}{1 - (-2)} \\ &= \frac{4 - 7}{3} \quad \begin{array}{l} g(1) = 3(1)^2 + 2(1) - 1 = 4 \\ g(-2) = 3(-2)^2 + 2(-2) - 1 = 7 \end{array} \\ &= -\frac{3}{3} = -1\end{aligned}$$

# Solution continued

---

(b) The slope of the secant line containing  $(-2, g(-2)) = (-2, 7)$  and  $(1, g(1)) = (1, 4)$  is  $m_{\text{sec}} = -1$ . Use the point-slope form to find an equation of the secant line.

$$y - y_1 = m_{\text{sec}}(x - x_1) \quad \text{Point-slope form of the secant line}$$

$$y - 7 = -1(x - (-2)) \quad x_1 = -2, y_1 = g(-2) = 7, m_{\text{sec}} = -1$$

$$y - 7 = -x - 2 \quad \text{Distribute.}$$

$$y = -x + 5 \quad \text{Slope-intercept form of the secant line}$$

# Solution continued

---

(c) The figure shows the graph of  $g$  along with the secant line  $y = -x + 5$ .

