

# 1

## Functions



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Moroccan runner Hicham El Guerrouj, current world record holder for the mile run, bested the record set 6 years earlier by 1.26 seconds.

# 1.4

# FUNCTIONS: POLYNOMIAL, RATIONAL, AND EXPONENTIAL



# Introduction

# Introduction

- In this section we will define other useful types of functions, including polynomial, rational, exponential, and logarithmic functions.



# Polynomial Functions

# Polynomial Functions

A **polynomial function** (or simply a **polynomial**) is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where  $n$  is a nonnegative integer and  $a_0, a_1, \dots, a_n$  are (real) numbers, called **coefficients**.

The *domain* of a polynomial is  $\mathbb{R}$ , the set of all (real) numbers.

# Polynomial Functions

The **degree** of a polynomial is the highest power of the variable. The following are polynomials.

$$f(x) = 2x^8 - 3x^7 + 4x^5 - 5$$

A polynomial of degree 8 (since the highest power of  $x$  is 8)

$$f(x) = -4x^2 - \frac{1}{3}x + 19$$

A polynomial of degree 2 (a quadratic function)

$$f(x) = x - 1$$

A polynomial of degree 1 (a linear function)

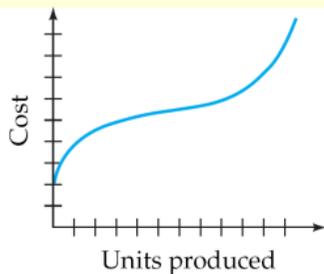
$$f(x) = 6$$

A polynomial of degree 0 (a constant function)

# Polynomial Functions

Polynomials are used to model many situations in which change occurs at different rates.

For example, the polynomial graphed on the right might represent the total cost of manufacturing  $x$  units of a product.



A cost function may increase at different rates at different production levels.

At first, costs rise steeply because of high start-up expenses, then more slowly as the economies of mass production come into play, and finally more steeply as new production facilities need to be built.

Polynomial equations can often be solved by factoring.

*important!!*



## Example 1 – SOLVING A POLYNOMIAL EQUATION BY FACTORING

$$\text{Solve } 3x^4 - 6x^3 = 24x^2$$

Solution:

$$3x^4 - 6x^3 - 24x^2 = 0$$

$$3x^2(x^2 - 2x - 8) = 0$$

$$3x^2 \cdot (x - 4) \cdot (x + 2) = 0$$

$\underbrace{\hspace{1.5cm}}$   $\underbrace{\hspace{1.5cm}}$   $\underbrace{\hspace{1.5cm}}$   
Equals zero at  $x = 0$    Equals zero at  $x = 4$    Equals zero at  $x = -2$

$$x = 0, \quad x = 4, \quad x = -2$$

Rewritten with all the terms on the left side

Factoring out  $3x^2$

Factoring further

Finding the zeros of each factor

Solutions



# Rational Functions

# Rational Functions

The word “ratio” means fraction or quotient, and the quotient of two polynomials is called a **rational function**.

The following are rational functions.

$$f(x) = \frac{3x + 2}{x - 2} \quad g(x) = \frac{1}{x^2 + 1}$$

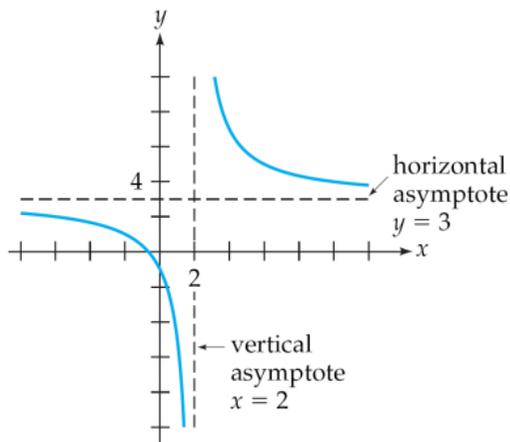
A rational function is a polynomial over a polynomial

The domain of a rational function is the set of numbers for which the denominator is not zero.

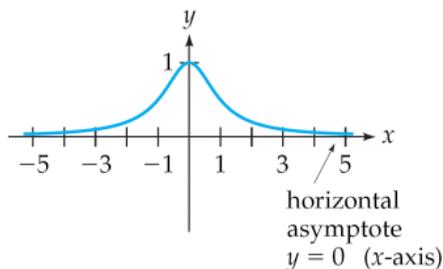
For example, the domain of the function  $f(x)$  on the left above is  $\{x \mid x \neq 2\}$  (since  $x = 2$  makes the denominator zero), and the domain of  $g(x)$  on the right is the set of all real numbers  $\mathbb{R}$  (since  $x^2 + 1$  is never zero).

# Rational Functions

The graphs of these functions are shown below. Notice that these graphs have **asymptotes**, lines that the graphs *approach* but never actually reach.



Graph of  $f(x) = \frac{3x + 2}{x - 2}$



Graph of  $g(x) = \frac{1}{x^2 + 1}$



# Exponential Functions

# Exponential Functions

A function in which the independent variable appears in the exponent, such as  $f(x) = 2^x$ , is called an **exponential function**.

## Example 2 – GRAPHING AN EXPONENTIAL FUNCTION

Graph the exponential function  $f(x) = 2^x$ .

**Solution:**

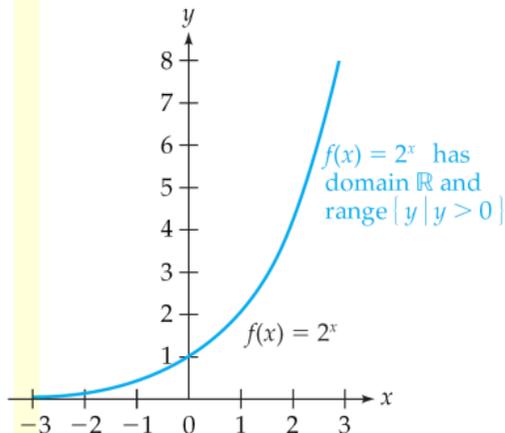
This function is defined for *all* real numbers, so its domain is  $\mathbb{R}$ .

## Example 2 – Solution

cont'd

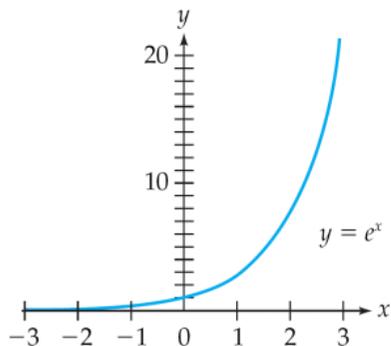
Values of the function are shown in the table and plotting these points and drawing a smooth curve through them gives the curve below.

$x$	$y = 2^x$
3	$2^3 = 8$
2	$2^2 = 4$
1	$2^1 = 2$
0	$2^0 = 1$
-1	$2^{-1} = \frac{1}{2}$
-2	$2^{-2} = \frac{1}{4}$
-3	$2^{-3} = \frac{1}{8}$

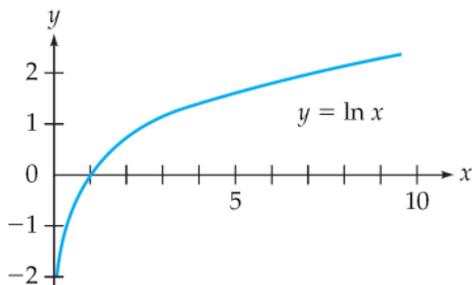


# Exponential Functions

An important function is the logarithmic function to the base  $e$ , written  $f(x) = \ln x$ . These functions are graphed below.



The exponential function  $e^x$  has domain  $\mathbb{R}$  and range  $\{y \mid y > 0\}$ .



The natural logarithm function has domain  $\{x \mid x > 0\}$  and range  $\mathbb{R}$ .



# Piecewise Linear Functions

# Piecewise Linear Functions

The rule for calculating the values of a function may be given in several parts.

If each part is linear, the function is called a **piecewise linear function**, and its graph consists of “pieces” of straight lines.

## Example 3 – GRAPHING A PIECEWISE LINEAR FUNCTION

This notation means:

$$\text{Graph } f(x) = \begin{cases} 5 - 2x & \text{if } x \geq 2 \\ x + 3 & \text{if } x < 2 \end{cases}$$

← Use the top formula for  $x \geq 2$   
← Use the bottom formula for  $x < 2$

**Solution:**

We graph one “piece” at a time.

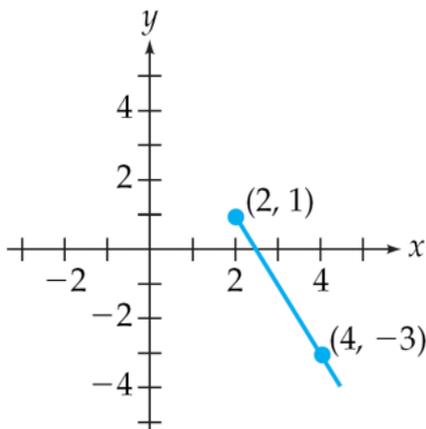
*Step 1:* To graph the first part,  $f(x) = 5 - 2x$  if  $x \geq 2$ , we use the “endpoint”  $x = 2$  and also  $x = 4$  (or any other  $x$ -value satisfying  $x \geq 2$ ).

## Example 3 – Solution

cont'd

The points are  $(2, 1)$  and  $(4, -3)$ , with the  $y$ -coordinates calculated from  $f(x) = 5 - 2x$ .

Draw the line through these two points, but only for  $x \geq 2$  (from  $x = 2$  to the *right*), as shown below.

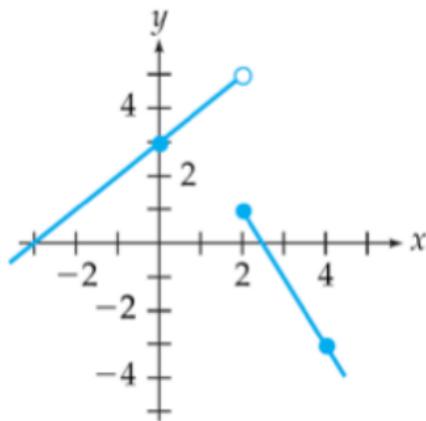


## Example 3 – Solution

cont'd

*Step 2:* For the second part,  $f(x) = x + 3$  if  $x < 2$ , the restriction  $x < 2$  means that the line ends just *before*  $x = 2$ .

We mark this “missing point”  $(2, 5)$  by an “open circle”  $\circ$  to indicate that it is *not* included in the graph (the  $y$ -coordinate comes from  $f(x) = x + 3$ ).



For a second point, choose  $x = 0$   
(or any other  $x < 2$ ), giving  $(0, 3)$ .

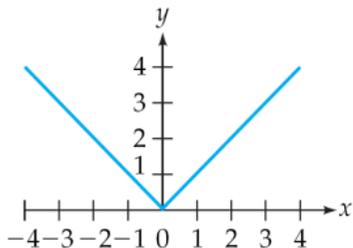
Draw the line through these two points,  
but only for  $x < 2$  (to the *left* of  $x = 2$ ),  
completing the graph of the function.

# Piecewise Linear Functions

An important piecewise linear function is the *absolute value* function.

## Absolute Value Function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



The absolute value function  $f(x) = |x|$  has a "corner" at the origin.



# Composite Functions

# Composite Functions

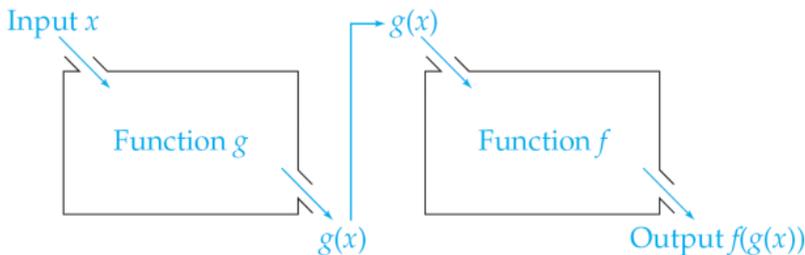
Just as we substitute a *number* into a function, we may substitute a *function* into a function. For two functions  $f$  and  $g$ , evaluating  $f$  at  $g(x)$  gives  $f(g(x))$ , called the *composition* of  $f$  with  $g$  evaluated at  $x$ .

## Composite Functions

The *composition* of  $f$  with  $g$  evaluated at  $x$  is  $f(g(x))$ .

# Composite Functions

The domain of  $f(g(x))$  is the set of all numbers  $x$  in the domain of  $g$  such that  $g(x)$  is in the domain of  $f$ . If we think of the functions  $f$  and  $g$  as “numerical machines,” then the composition  $f(g(x))$  may be thought of as a *combined* machine in which the output of  $g$  is connected to the input of  $f$ .



A “machine” for generating the composition of  $f$  with  $g$ .

A number  $x$  is fed into the function  $g$ , and the output  $g(x)$  is then fed into the function  $f$ , resulting in  $f(g(x))$ .

## Example 6 – FINDING COMPOSITE FUNCTIONS

If  $f(x) = x^7$  and  $g(x) = x^3 - 2x$ , find:

a.  $f(g(x))$

b.  $f(f(x))$

Solution:

$$\begin{array}{l} \text{a. } f(g(x)) = [g(x)]^7 = (x^3 - 2x)^7 \\ \underbrace{\hspace{10em}}_{f(x) = x^7 \text{ with } x \text{ replaced by } g(x)} \quad \bigg| \quad \underbrace{\hspace{10em}}_{\text{Using } g(x) = x^3 - 2x} \end{array}$$

## Example 6 – Solution

cont'd

$$\begin{aligned} \text{b. } f(f(x)) &= [f(x)]^7 = (x^7)^7 = x^{49} \\ &\quad \underbrace{\hspace{1.5cm}}_{f(x) = x^7 \text{ with } x \text{ replaced by } f(x)} \quad \underbrace{\hspace{1.5cm}}_{\text{Using } f(x) = x^7} \end{aligned}$$



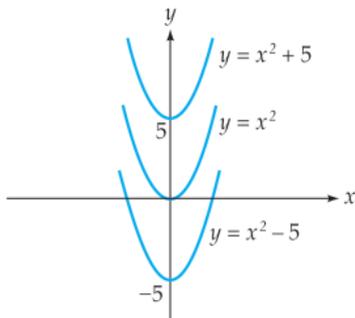
# Shifts of Graphs

# Shifts of Graphs

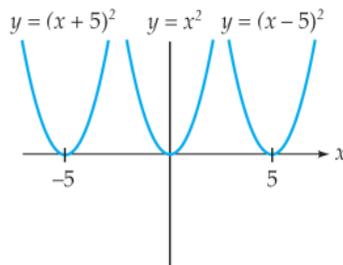
Sometimes the graph of a composite function is just a horizontal or vertical shift of an original graph. This occurs when one of the functions is simply the addition or subtraction of a constant.

The following diagram shows the graph of  $y = x^2$  together with various shifts and the functions that generate them.

Vertical shifts



Horizontal shifts



# Shifts of Graphs

In general, adding to or subtracting from the *x-value* means a *horizontal* shift, while adding to or subtracting from the *function* means a *vertical* shift.

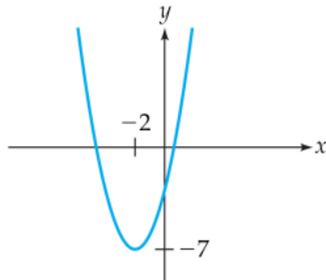
These same ideas hold for *any* function: given the graph of  $y = f(x)$ , adding or subtracting a positive number  $a$  to the function  $f(x)$  or to the variable  $x$  shifts the graph as follows:

## Shifts of Graphs

<u>Function</u>	<u>Shift</u>	
$y = f(x) + a$	shifted <i>up</i> by $a$ units	} Vertical shifts
$y = f(x) - a$	shifted <i>down</i> by $a$ units	
$y = f(x + a)$	shifted <i>left</i> by $a$ units	} Horizontal shifts
$y = f(x - a)$	shifted <i>right</i> by $a$ units	

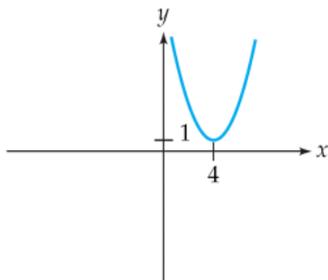
# Shifts of Graphs

A graph can be shifted both horizontally and vertically, as illustrated by the following shifts of  $y = x^2$ :



$$y = (x + 2)^2 - 7$$

(shifted *left* 2 units  
and *down* 7 units)



$$y = (x - 4)^2 + 1$$

(shifted *right* 4 units  
and *up* 1 unit)

Such double shifts can be applied to *any* function

$y = f(x)$ : the graph of  $y = f(x + a) + b$  is shifted *left*  $a$  units and *up*  $b$  units (with the understanding that a *negative*  $a$  or  $b$  means that the direction is reversed).

## Example 9 – FINDING $f(x + h)$ FROM $f(x)$

If  $f(x) = x^2 - 5x$ , find  $f(x + h)$ :

Solution:

$$\begin{aligned} f(x + h) &= \underbrace{(x + h)^2}_{\downarrow} - \underbrace{5(x + h)}_{\downarrow} \\ &= \underbrace{x^2 + 2xh + h^2}_{\text{Expanding}} - \underbrace{5x - 5h} \end{aligned}$$

$f(x) = x^2 - 5x$  with each  $x$   
replaced by  $x + h$

Expanding



# Difference Quotients

# Difference Quotients

The quantity  $\frac{f(x+h) - f(x)}{h}$  will be very important when

we begin studying calculus. It is called the **difference quotient**, since it is a quotient whose numerator is a difference.

*important!*

It gives the slope (rise over run) between the points in the curve  $y = f(x)$  at  $x$  and at  $x + h$ .

## Example 10 – FINDING A DIFFERENCE QUOTIENT

If  $f(x) = x^2 - 4x + 1$ , find and simplify  $\frac{f(x + h) - f(x)}{h}$  ( $h \neq 0$ )

Solution:

$$\begin{aligned}\frac{f(x + h) - f(x)}{h} &= \frac{\overbrace{(x + h)^2 - 4(x + h) + 1}^{f(x + h)} - \overbrace{(x^2 - 4x + 1)}^{f(x)}}{h} \\ &= \frac{x^2 + 2xh + h^2 - 4x - 4h + 1 - x^2 + 4x - 1}{h} && \text{Expanding} \\ &= \frac{x^2 + 2xh + h^2 - \cancel{4x} - 4h + \cancel{1} - \cancel{x^2} + \cancel{4x} - \cancel{1}}{h} && \text{Canceling}\end{aligned}$$

## Example 10 – FINDING A DIFFERENCE QUOTIENT cont'd

$$= \frac{2xh + h^2 - 4h}{h} = \frac{h(2x + h - 4)}{h}$$

Factoring an  $h$   
from the top

$$= \frac{h(2x + h - 4)}{h} = 2x + h - 4$$

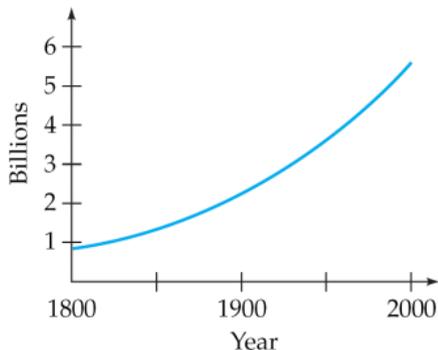
Canceling  $h$  from  
top and bottom  
(since  $h \neq 0$ )



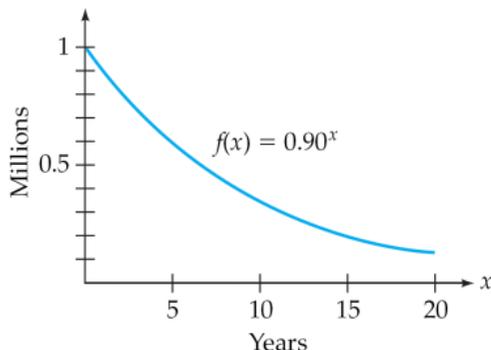
# Exponential Regression (Optional)

# Exponential Regression (Optional)

If data appear to lie along an *exponential* curve, as shown below, we may fit an exponential curve to the data using **exponential regression**. 



World population since the year 1800 can be approximated by an exponential function.



A population of 1 million that declines by 10% each year is modeled by an exponential function.

# 4

## Exponential and Logarithmic Functions



James King/Alamy/Photo Researchers, Inc.

## 4.3

# DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS



# Introduction

# Introduction

- In this section we differentiate these new functions and use their derivatives for graphing, optimization, and finding rates of change.
- We emphasize *natural* (base  $e$ ) logs and exponentials, since most applications use these exclusively.



# Derivatives of Logarithmic Functions

# Derivatives of Logarithmic Functions

The rule for differentiating the natural logarithm function is as follows:

Derivative of  $\ln x$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

The derivative of  $\ln x$  is 1 over  $x$

## Example 1 – DIFFERENTIATING A LOGARITHMIC FUNCTION

Differentiate  $f(x) = x^3 \ln x$ .

**Solution:**

The function is a *product*,  $x^3$  times  $\ln x$ , so we use the Product Rule.

$$\frac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2$$

Derivative of the first      Second left alone      First left alone      Derivative of  $\ln x$

From  $x^3 \frac{1}{x} = x^2$

# Derivatives of Logarithmic Functions

The preceding rule, together with the Chain Rule, shows how to differentiate the natural logarithm of a *function*. For any differentiable function  $f(x)$  that is positive:

Derivative of  $\ln f(x)$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

The derivative of the natural log of a function is the derivative of the function over the function

Notice that the right-hand side does not involve logarithms at all.

## Example 2 – DIFFERENTIATING A LOGARITHMIC FUNCTION

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1} \text{thout the ln}$$



# Derivatives of Exponential Functions

# Derivatives of Exponential Functions

The rule for differentiating the exponential function  $e^x$  is as follows:

Derivative of  $e^x$

$$\frac{d}{dx} e^x = e^x$$

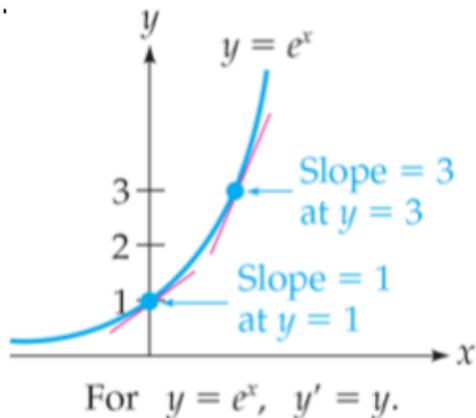
The derivative of  $e^x$  is simply  $e^x$

The function  $e^x$  is unchanged by the operation of differentiation.

# Derivatives of Exponential Functions

This rule can be interpreted graphically:  
if  $y = e^x$ , then  $y' = e^x$ , so that  $y = y'$ .

This means that on the graph of  $y = e^x$ , the slope  $y'$  always equals the  $y$ -coordinate, as shown in the graph on the right.



Since  $y'$  and  $y''$  both equal  $e^x$ , they are always positive and the graph is always increasing and concave upwards.

## Example 4 – FINDING A DERIVATIVE INVOLVING $e^x$

Find  $\frac{d}{dx} \left( \frac{e^x}{x} \right)$ .

**Solution:**

Since the function is a quotient, we use the Quotient Rule:

$$\frac{d}{dx} \left( \frac{e^x}{x} \right) = \frac{x \cdot e^x - 1 \cdot e^x}{x^2} = \frac{xe^x - e^x}{x^2}$$



# Derivatives of Exponential Functions

The rule for differentiating  $ex$ , together with the Chain Rule, shows how to differentiate  $ef(x)$ .

For any differentiable function  $f(x)$ :



Derivative of  $e^{f(x)}$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$$

The derivative of  $e$  to a function is  $e$  to the function times the derivative of the function

That is, to differentiate  $ef(x)$  we simply “copy” the original  $ef(x)$  and then by the derivative of the exponent.

## Example 6 – DIFFERENTIATING AN EXPONENTIAL FUNCTION

$$\frac{d}{dx} e^{x^4+1} = \underbrace{e^{x^4+1}}_{\text{Copied}} (4x^3)$$

Reversing the order

Derivative of the exponent

$$= 4x^3 e^{x^4+1}$$

# Derivatives of Exponential Functions

The formulas for differentiating natural logarithmic and exponential functions are summarized as follows, with  $f(x)$  written simply as  $f$ .

## *Logarithmic Formulas*

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln f = \frac{f'}{f}$$

## *Exponential Formulas*

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^f = e^f \cdot f'$$

Top formulas apply only to  $\ln x$  and  $e^x$

Bottom formulas apply to  $\ln$  and  $e$  of a *function*

## Example 8 – DIFFERENTIATING A LOGARITHMIC AND EXPONENTIAL FUNCTION

Find the derivative of  $\ln(1 + e^x)$ .

**Solution:**

$$\frac{d}{dx} \ln(1 + e^x) = \frac{\frac{d}{dx}(1 + e^x)}{1 + e^x} = \frac{e^x}{1 + e^x}$$

Using  $\frac{d}{dx} \ln f = \frac{f'}{f}$       Working out the numerator

# Derivatives of Exponential Functions

Derivative of  $e^{kx}$

$$\frac{d}{dx} e^{kx} = ke^{kx}$$

For any constant  $k$

This formula says that the rate of change (the derivative) of  $e^{kx}$  is proportional to itself. That is, the function satisfies the *differential equation*

$$y' = ky$$

We noted this earlier when we observed that in exponential growth a quantity *grows in proportion to itself*.

# Derivatives of Exponential Functions

These differentiation formulas enable us to find instantaneous rates of change of logarithmic and exponential functions.

In many applications the variable stands for time, so we use  $t$  instead of  $x$ .

## Example 9 – FINDING A RATE OF IMPROVEMENT OF A SKILL

After  $t$  weeks of practice a pole vaulter can vault

$$H(t) = 15 - 11e^{-0.1t}$$

feet. Find the rate of change of the athlete's jumps after

a. 0 weeks (at the beginning of training)

b. 12 weeks

*Solution:*

We differentiate to find the rate of change

$$H'(t) = -11(-0.1)e^{-0.1t} = \underbrace{1.1e^{-0.1t}}_{\text{Simplifying}}$$

Using  $\frac{d}{dt} e^{kt} = ke^{kt}$

Differentiating  
 $15 - 11e^{-0.1t}$

## Example 9 – Solution

cont'd

a. For the rate of change after 0 weeks:

$$H'(0) = 1.1e^{-0.1(0)} = 1.1e^0 = 1.1$$

$$H'(t) = 1.1e^{-0.1t} \text{ with } t = 0$$

b. After 12 weeks:

$$\begin{aligned} H'(12) &= 1.1e^{-0.1(12)} \\ &= 1.1e^{-1.2} \approx 1.1(0.30) \\ &= 0.33 \end{aligned}$$

$$H'(t) = 1.1e^{-0.1t} \text{ with } t = 12$$

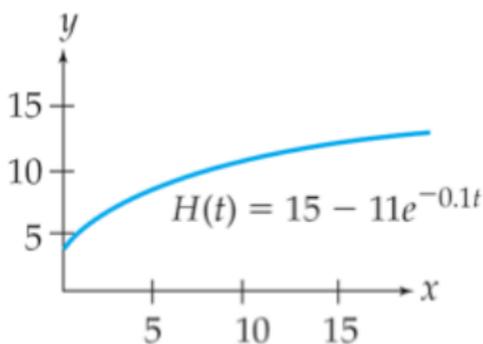
Using a calculator

At first, the vaults increased by 1.1 feet per week. After 12 weeks, the gain was only 0.33 foot (about 4 inches) per week.

# Derivatives of Exponential Functions

This result is typical of learning a new skill: early improvement is rapid, later improvement is slower.

This trend is called *diminishing returns*, and may be seen in the leveling off of the polevault heights in the graph below.





# Maximizing Consumer Expenditure

# Maximizing Consumer Expenditure

The amount of a commodity that consumers will buy depends on the price of the commodity.

For a commodity whose price is  $p$ , let the consumer demand be given by a function  $D(p)$ . Multiplying the number of units  $D(p)$  by the price  $p$  gives the total *consumer expenditure* for the commodity.

## Consumer Demand and Expenditure

Let  $D(p)$  be the consumer demand at price  $p$ . Then the consumer expenditure is

$$E(p) = p \cdot D(p)$$

## Example 10 – MAXIMIZING CONSUMER EXPENDITURE

If consumer demand for a commodity is  $D(p) = 10,000e^{-0.02p}$  units per week, where  $p$  is the selling price, find the price that maximizes consumer expenditure.

### *Solution:*

Using the preceding formula for consumer expenditure,

$$E(p) = p \cdot 10,000 e^{-0.02p} \qquad E(p) = p \cdot D(p)$$

$$= 10,000pe^{-0.02p}$$

To maximize  $E(p)$  we differentiate:

$$E'(p) = \underbrace{\text{Derivative of } 10,000p}_{-0.02p + 10,000} \cdot \underbrace{\text{Derivative of } e^{-0.02p}}_{-0.02e^{-0.02p}} \qquad \text{Using the Product Rule to differentiate } E(p) = 10,000p \cdot e^{-0.02p}$$

## Example 10 – Solution

cont'd

$$= 10,000e^{-0.02p} - 200pe^{-0.02p}$$

$$= 200e^{-0.02p}(50 - p)$$

$$\text{CN: } p = 50$$

Simplifyin

g  
Factorin

g

Critical number from  
(50 - p) (since e to a  
power is never zero)

We calculate  $E''$  for the second-derivative test:

$$E''(p) = 200(-0.02)e^{-0.02p}(50 - p) + 200e^{-0.02p}(-1)$$

From  $E'(p) =$

$200e^{-0.02p} \cdot (50 - p)$   
using the

Product Rule

Simplifying

$$= -4e^{-0.02p}(50 - p) - 200e^{-0.02p}$$

## Example 10 – Solution

cont'd

At the critical number  $p = 50$ ,

$$E''(50) = -4e^{-0.02(50)}(50 - 50) - 200e^{-0.02(50)} \quad \text{Substituting } p = 50$$

$$= -200 - \frac{200}{e} \quad \text{Simplifying}$$

$E''$  is negative, so the expenditure  $E(p)$  is maximized at  $p = 50$ :

Consumer expenditure is maximized at price \$50.



# Graphing Logarithmic and Exponential Functions

## Graphing Logarithmic and Exponential Functions

To graph logarithmic and exponential functions using a graphing calculator, we first find critical points and possible inflection points, and then graph the function on a window including these points.

(If graphing “by hand,” we would make sign diagrams for the first and second derivatives and then sketch the graph).

## Example 11 – GRAPHING AN EXPONENTIAL FUNCTION

Graph  $f(x) = e^{-x^2/2}$ .

**Solution:**

As before, we write the function as  $f(x) = e^{-\frac{1}{2}x^2}$ .

The derivative is

$$f'(x) = e^{-\frac{1}{2}x^2}(-x) = -xe^{-\frac{1}{2}x^2} \quad \text{Using } \frac{d}{dx} e^f = e^f \cdot f'$$

Derivative of the exponent

$$\text{CN: } x = 0$$

Critical number is 0

$$y = 1$$

From  $y =$

evaluated at  $x = 0$   
 $e^{-\frac{1}{2}x^2}$

The second derivative is

$$f''(x) = (-1)e^{-\frac{1}{2}x^2} - xe^{-\frac{1}{2}x^2}(-x) \quad \text{From } f'(x) = -x \cdot e^{-\frac{1}{2}x^2} \text{ using the Product Rule}$$

# Example 11 – Solution

cont'd

$$= -e^{-\frac{1}{2}x^2} + x^2 e^{-\frac{1}{2}x^2}$$

$$= e^{-\frac{1}{2}x^2}(-1 + x^2)$$

$$= (x^2 - 1)e^{-\frac{1}{2}x^2}$$

$$= (x + 1)(x - 1)e^{-\frac{1}{2}x^2}$$

$$x = \pm 1 \quad \text{Where } f' = 0$$

$$y = e^{-\frac{1}{2}} \approx 0.6$$

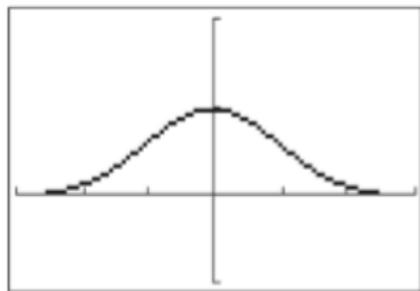
From  $y = e^{-\frac{1}{2}x^2}$   
evaluated at  $x = \pm 1$

## Graphing Logarithmic and Exponential Functions

Based on these values, we choose the graphing window as follows.

For the  $x$ -values we choose  $[-3, 3]$  (to include 0 and  $\pm 1$  and beyond), and for the  $y$ -values we choose  $[-1, 2]$  (to include 1 and 0.6 and above and below).

This window gives the graph on the right.



$$f(x) = e^{-x^2/2} \text{ on } [-3, 3] \text{ by } [-1, 2]$$

(Many other windows would be just as good, and after seeing the graph you might want to adjust the window.)

# 4

## Exponential and Logarithmic Functions



James King/Alamy/Photo Researchers, Inc.

## 4.4

## TWO APPLICATIONS TO ECONOMICS: RELATIVE RATES AND ELASTICITY OF DEMAND



# Introduction

# Introduction

- In this section we define **relative rates of change** and see how they are used in economics.
- We then define the very important economic concept of **elasticity of demand**.



# Relative Versus Absolute Rates

# Relative Versus Absolute Rates

The derivative of a function gives its rate of change.

For example, if  $f(t)$  is the cost of a pair of shoes at time  $t$  years, then  $f'(t)$  is the rate of change of cost (in dollars per year). That is,  $f' = 3$  would mean that the price of shoes is increasing at the rate of \$3 per year.

Similarly, if  $g(t)$  is the price of a new automobile at time  $t$  years, then  $g' = 300$  would mean that automobile prices are increasing at the rate of \$300 per year

# Relative Versus Absolute Rates

Does this mean that car prices are rising 100 times as fast as shoe prices? In absolute terms, yes.

However, this does not take into account the enormous price difference between automobiles and shoes.



# Relative Rates of Change

# Relative Rates of Change

If  $f(t)$  is the price of an item at time  $t$ , then the rate of change is  $f'(t)$ , and the *relative* rate of change is  $f'(t)/f(t)$ , the derivative divided by the function.

We will sometimes call the derivative  $f'(x)$  the “absolute” rate of change to distinguish it from the relative rate of change  $f'(x)/f(x)$ .

# Relative Rates of Change

Relative rates are often more meaningful than absolute rates.

For example, it is easier to grasp the fact that the gross domestic product is growing at the relative rate of 3% a year than that it is growing at the absolute rate of \$400,000,000,000 per year.

# Relative Rates of Change

The expression  $f'(x)/f(x)$  is the derivative of the natural logarithm of  $f(x)$ :

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

This provides an alternative expression for the relative rate of change, in terms of logarithms.

## Relative Rate of Change

$$\left( \text{Relative rate of change of } f(t) \right) = \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}$$

For a differentiable function  $f > 0$

# Relative Rates of Change

We use the variable  $t$  since it often stands for time. Both formulas in the box give the same result, and the middle expression is sometimes called the *logarithmic derivative*, since it is found by first taking the logarithm and then the derivative.

The relative rate of change, being a ratio or a percent, does not depend on the units of the function.

Therefore, relative rates can be compared between different products, and even between different nations.

# Relative Rates of Change

This is in contrast to absolute rates of change (that is, derivatives), which *do* depend on the units (for example, dollars per year).

## Example 1 – FINDING A RELATIVE RATE OF CHANGE

If the gross domestic product  $t$  years from now is predicted to be  $G(t) = 8.2e^{\sqrt{t}}$  trillion dollars, find the relative rate of change 25 years from now.

We give two solutions, showing the use of both formulas.

**Solution:** (using the  $\frac{d}{dt} \ln f(t)$  formula)

First we simplify:

$$\ln G(t) = \ln 8.2 e^{\sqrt{t}} \quad \text{Taking natural logs}$$

$$= \ln 8.2 + \ln e^{\sqrt{t}} \quad \text{Log of a product is the sum of the logs}$$

# Example 1 – Solution

cont'

d

$$\begin{aligned} &= \ln 8.2 + \sqrt{t} \\ &= \ln 8.2 + t^{1/2} \end{aligned}$$

$$\ln e^{\sqrt{t}} = \sqrt{t}$$

Then we differentiate:

$$\begin{aligned} \frac{d}{dt} (\ln 8.2 + t^{1/2}) &= 0 + \frac{1}{2} t^{-1/2} \quad \begin{array}{l} \text{a constant, so} \\ \text{its derivative is zero} \end{array} \\ &= \frac{1}{2} t^{-1/2} \end{aligned}$$

Finally, we evaluate at the given time  $t = 25$ :

$$\begin{aligned} \frac{1}{2} (25)^{-1/2} &= \frac{1}{2} \frac{1}{\sqrt{25}} \quad \frac{1}{2} t^{-1/2} \text{ evaluated at } t = 25 \\ &= \frac{1}{2} \frac{1}{5} = \frac{1}{10} = 0.10 \end{aligned}$$

# Example 1 – Solution

cont'

d

Therefore, in 25 years the gross domestic product will be increasing at the relative rate of 0.10, or 10%, per year.

Alternative Solution : (using the  $\frac{f'(t)}{f(t)}$  formula)

$$G(t) = 8.2e^{\sqrt{t}} = 8.2e^{t^{1/2}}$$

Writing  $G(t)$  with  
fractional exponents

$$G'(t) = 8.2e^{t^{1/2}} \left( \underbrace{\frac{1}{2} t^{-1/2}} \right)$$

Differentiating

Derivative of the exponent

# Example 1 – Solution

cont'

d

Therefore, the relative rate of change  $\frac{G'(t)}{G(t)}$  is

$$\frac{G'(t)}{G(t)} = \frac{8.2e^{t^{1/2}} \left( \frac{1}{2} t^{-1/2} \right)}{8.2e^{t^{1/2}}} = \frac{1}{2} t^{-1/2}$$

Same result as with the first formula

At  $t = 25$

$$\frac{1}{2} (25)^{-1/2} = \frac{1}{2} \frac{1}{\sqrt{25}}$$

Again the same

$$= \frac{1}{2} \frac{1}{5} = \frac{1}{10} = 0.10$$

Therefore, the relative rate of change is 10%, just as we found before.



# Elasticity of Demand

# Elasticity of Demand

Farmers are aware of the paradox that an abundant harvest usually brings *lower* total revenue than a poor harvest.

The reason is simply that the larger quantities in an abundant harvest result in lower prices, which in turn cause increased demand, but the demand does not increase enough to compensate for the lower prices.

Revenue is price times quantity,  $R = p \cdot q$ , and when one of these quantities rises, the other falls.

# Elasticity of Demand

The question is whether the rise in one is enough to compensate for the fall in the other.

For example, if a 1% price decrease brings a 2% quantity increase, revenue will rise, but if the 1% price decrease brings only a  $\frac{1}{2}$ % quantity increase, revenue will fall.

The concept of *elasticity of demand* was invented to analyze such problems.

# Elasticity of Demand

Roughly speaking, we may think of elasticity as *the percentage change in demand divided by the percentage change in price*:

## Understanding Elasticity of Demand

$$E = \frac{\text{Percent change in demand}}{\text{Percent change in price}}$$

## Brief Examples

If a 1% change in price brings a 2% change in demand:

$$E = \frac{2\%}{1\%} = 2$$

If a 1% change in price brings only a  $\frac{1}{2}\%$  change in demand:

$$E = \frac{\frac{1}{2}\%}{1\%} = \frac{1}{2}$$

# Elasticity of Demand

We classify demand as elastic or inelastic depending on whether elasticity is greater than 1 or less than 1:

Demand is *elastic* if  $E > 1$

Demand is *inelastic* if  $E < 1$

Demand is *unit-elastic* if  $E = 1$

In the first Brief Example above,  $E = 2$ , so demand was *elastic*, while in the second example,  $E = \frac{1}{2}$ , so demand was *inelastic*.

# Elasticity of Demand

Intuitively, we may think of elasticity of demand as measuring how *responsive* demand is to price changes: *elastic* means *responsive* and *inelastic* means *unresponsive*.

That is, for elastic demand, a price cut will bring a large increase in demand, so total revenue will rise.

On the other hand, for inelastic demand, a price cut will bring only a slight increase in demand, so total revenue will fall.

# Elasticity of Demand

Economists calculate elasticity of demand for many products, and some typical elasticities are shown in the table.

Notice that for *necessities* (clothing, food), demand is inelastic since consumers need them even if prices rise, while for luxuries (restaurant meals) demand is elastic since consumers can cut back or find substitutes in response to price increases.

Good or Service	Elasticity
Clothing	0.20
Housing	0.30
Gasoline	0.43
Movies	0.87
Automobiles	1.87
Restaurant meals	2.27
Fresh fruit	3.02

Source: Houthaker and Taylor, *Consumer Demand in the United States, Review of Economics and Statistics*, 62

# Elasticity of Demand

## Demand Function

The demand function

$$x = D(p)$$

gives the quantity  $x$  of an item that will be demanded by consumers if the price is  $p$ .

Since, in general, demand falls as prices rise, the slope of the demand function is negative, as shown in the graph.

This is known as the *law of downward sloping demand*.



Law of downward-sloping demand



# Calculating Elasticity of Demand

# Calculating Elasticity of Demand

## Elasticity of Demand

For a demand function  $D(p)$ , the elasticity of demand is

$$E(p) = \frac{-p \cdot D'(p)}{D(p)}$$

Demand is *elastic* if  $E(p) > 1$  and *inelastic* if  $E(p) < 1$ .

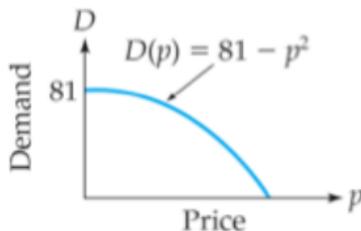
Elasticity, being composed of *relative* rates of change, does not depend on the units of the demand function.

Therefore, elasticities can be compared between different products, and even between different countries.

## Example 2 – FINDING ELASTICITY OF DEMAND FOR COMMUTER BUS SERVICE

A bus line estimates the demand function for its daily commuter tickets to be  $D(p) = 81 - p^2$  (in thousands of tickets), where  $p$  is the price in dollars ( $0 \leq p \leq 9$ ). Find the elasticity of demand when the price is:

- a. \$3    b. \$6



## Example 2 – Solution

$$\text{Defin } E(p) = \frac{-pD'(p)}{D(p)}$$

$$= \frac{-p(-2p)}{81 - p^2}$$

Substituting  $D(p) = 81 - p^2$   
so  $D'(p) = -2p$

$$= \frac{2p^2}{81 - p^2} \text{ plifying}$$

## Example 2 – Solution

cont'

d

a. Evaluating at  $p = 3$  gives

$$E(3) = \frac{2(3)^2}{81 - (3)^2} \quad E(p) = \frac{2p^2}{81 - p^2} \quad \text{with } p = 3$$

$$= \frac{18}{81 - 9} = \frac{18}{72} = \frac{1}{4}$$

*Interpretation:* The elasticity is less than 1, so demand for tickets is *inelastic* at a price of \$3. This means that a small price change (up or down from this level) will cause only a *slight* change in demand. More precisely, elasticity of  $\frac{1}{4}$  means that a 1% price change will cause only about a  $\frac{1}{4}\%$  change in demand.

## Example 2 – Solution

cont'

d

b. At the price of \$6, the elasticity of demand is

$$\begin{aligned} E(6) &= \frac{2(6)^2}{81 - (6)^2} & E(p) &= \frac{2p^2}{81 - p^2} \text{ with } p = 6 \\ &= \frac{72}{81 - 36} = \frac{8}{5} = 1.6 \end{aligned}$$

*Interpretation:* The elasticity is greater than 1, so demand is *elastic* at a price of \$6. This means that a small change in price (up or down from this level) will cause a relatively *large* change in demand. In particular, an elasticity of 1.6 means that a price change of 1% will cause about a 1.6% change in demand.



# Using Elasticity to Increase Revenue

# Using Elasticity to Increase Revenue

In Example 2 we found that at a price of \$3, demand is inelastic ( $E = \frac{1}{4} < 1$ ); and so demand responds only *weakly* to price changes.

Therefore, to increase revenue the company should *raise* prices, since the higher prices will drive away only a relatively small number of customers.

On the other hand, at a price of \$6, demand is elastic ( $E = 1.6 > 1$ ), and so demand is very responsive to price changes. In this case, to increase revenue the company should *lower* prices, since this will attract more than enough new customers to compensate for the price decrease.

# Using Elasticity to Increase Revenue

In general:

## Elasticity and Revenue

To increase revenue:

If demand is elastic ( $E > 1$ ), you should *lower* prices.

If demand is inelastic ( $E < 1$ ), you should *raise* prices.

This should be intuitively clear: In simplest terms, you should *lower* prices if demand *will* change and *raise* prices if demand *won't* change.

# Using Elasticity to Increase Revenue

This statement shows why elasticity of demand is important to any company that cuts prices in an attempt to boost revenue, or to any utility that raises prices in order to increase revenue.

Elasticity shows whether the strategy will succeed or fail.

The borderline case,  $E = 1$  (called *unit-elasticity*), is where revenue cannot be raised, which will be the case if revenue is at its maximum.

Therefore, elasticity must be unitary when revenue is maximized.

# Using Elasticity to Increase Revenue

At maximum revenue, elasticity of demand must equal 1.

We could use this fact as a basis for a new method for maximizing revenue, but instead, we will stick with our earlier (and easier) method of maximizing functions by finding critical numbers.