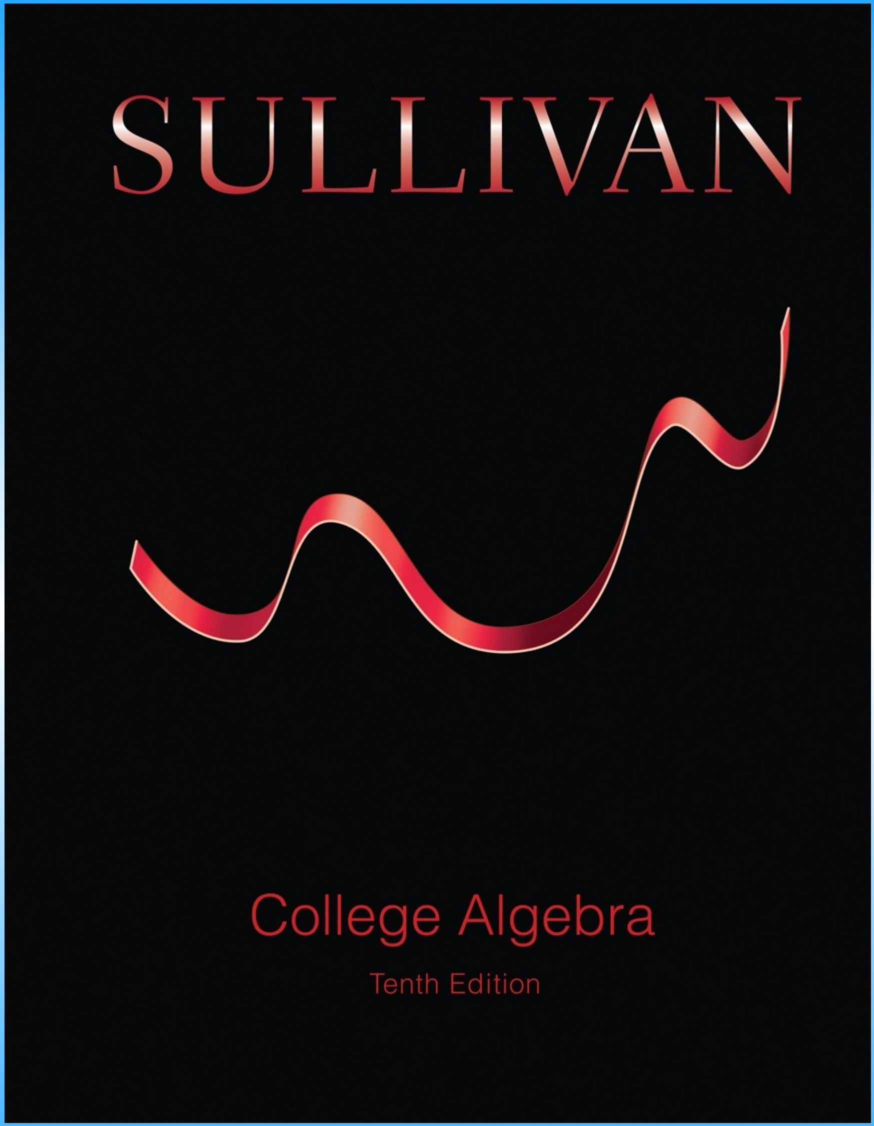
Chapter 3

Section 1



3.1 Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Section 1.5, pp. 120–121)
- Solving Inequalities (Section 1.5, pp. 123–126)
- Evaluating Algebraic Expressions, Domain of a Variable (Chapter R, Section R.2, pp. 20–23)
- Rationalizing Denominators (Chapter R, Section R.8, p. 75)

Now Work the **'Are You Prepared?'** problems on page 210.

OBJECTIVES

1. Determine Whether a Relation Represents a Function (p. 199)
2. Find the Value of a Function (p. 202)
3. Find the Difference Quotient of a Function (p. 205)
4. Find the Domain of a Function Defined by an Equation (p. 206)
5. Form the Sum, Difference, Product, and Quotient of Two Functions (p. 208)

Determine Whether a Relation Represents a Function

Figure: y = 3x -1

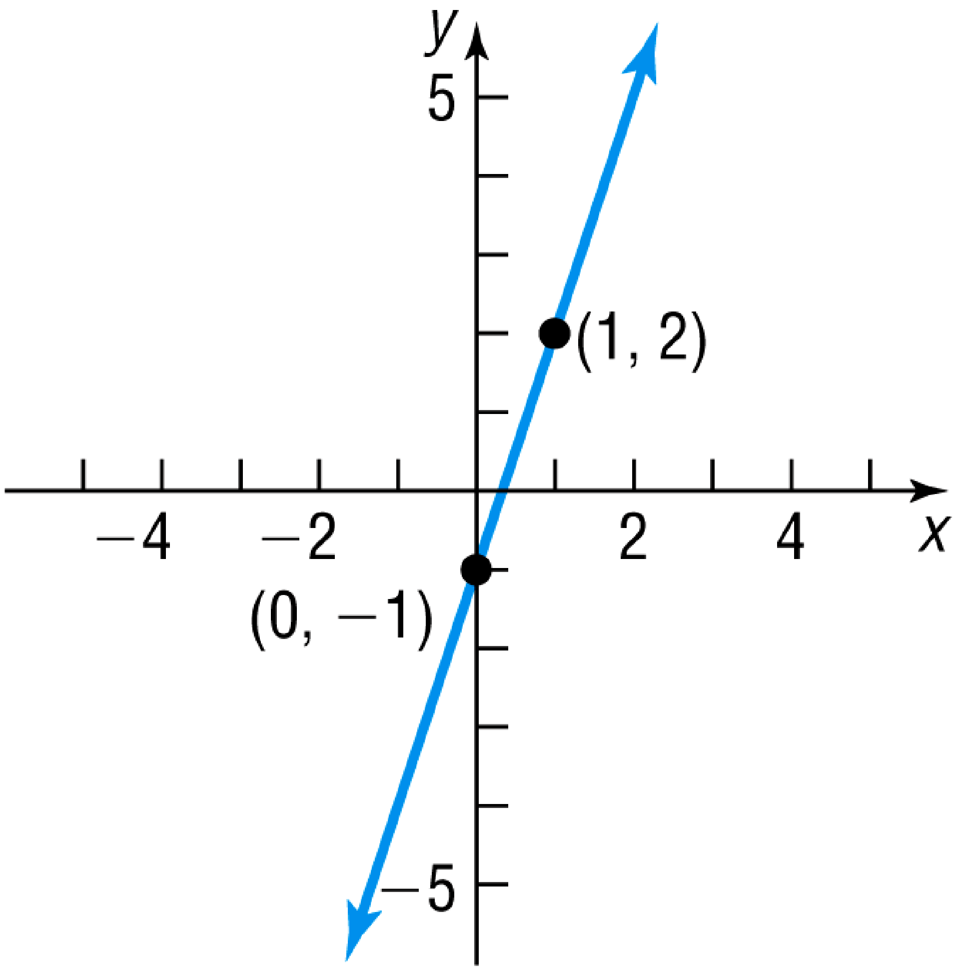


Figure: Number of Representatives

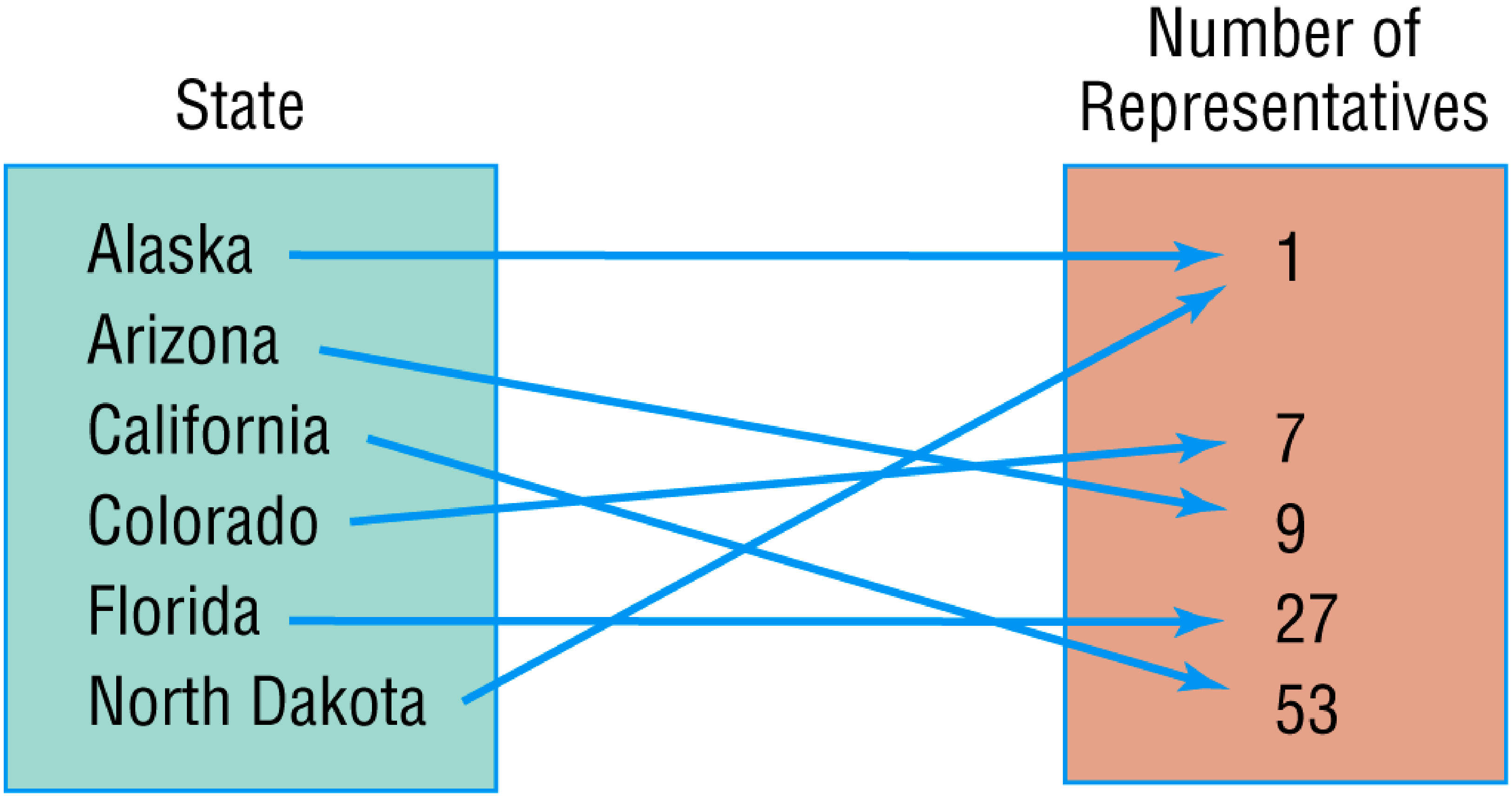


Figure: Phone Numbers

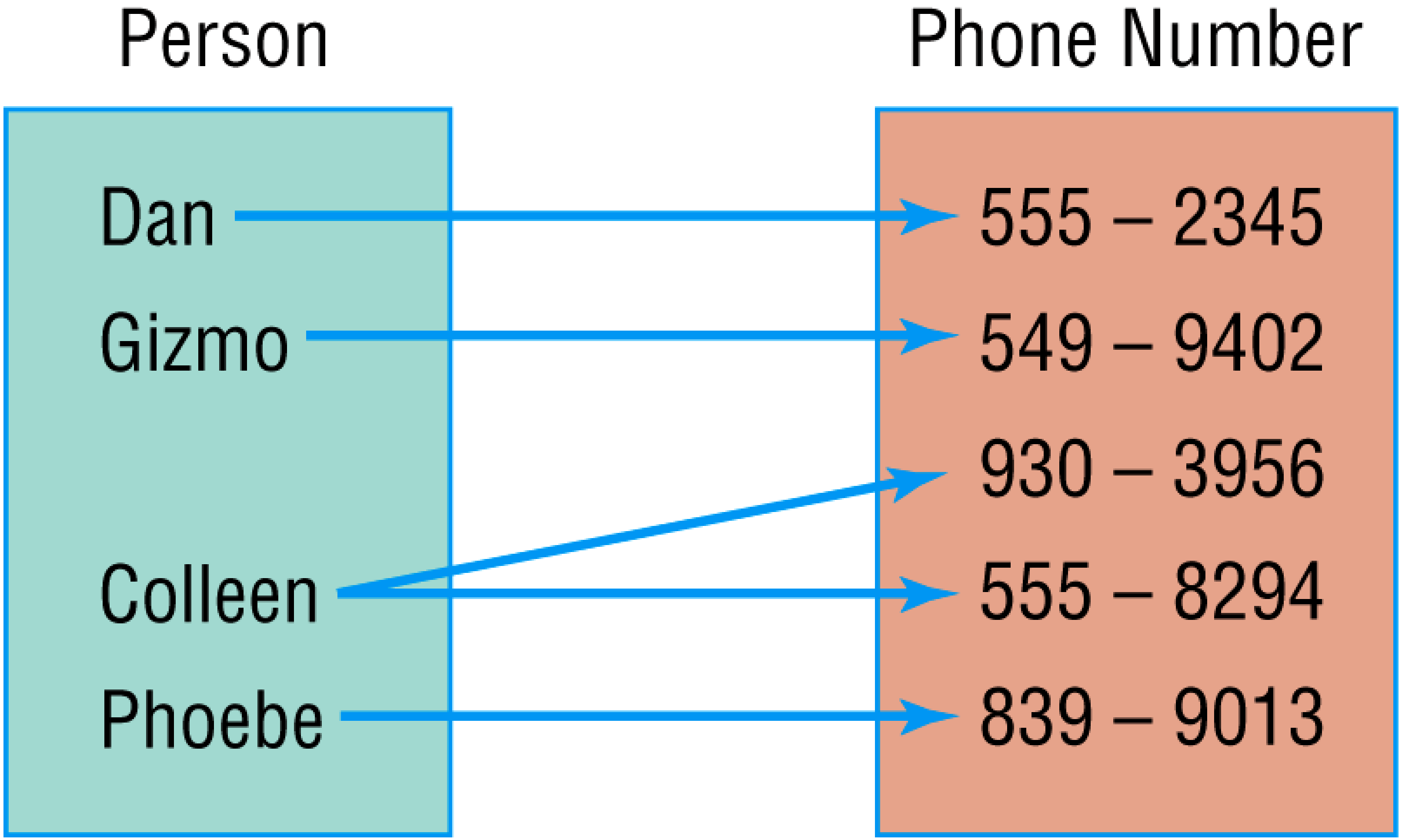
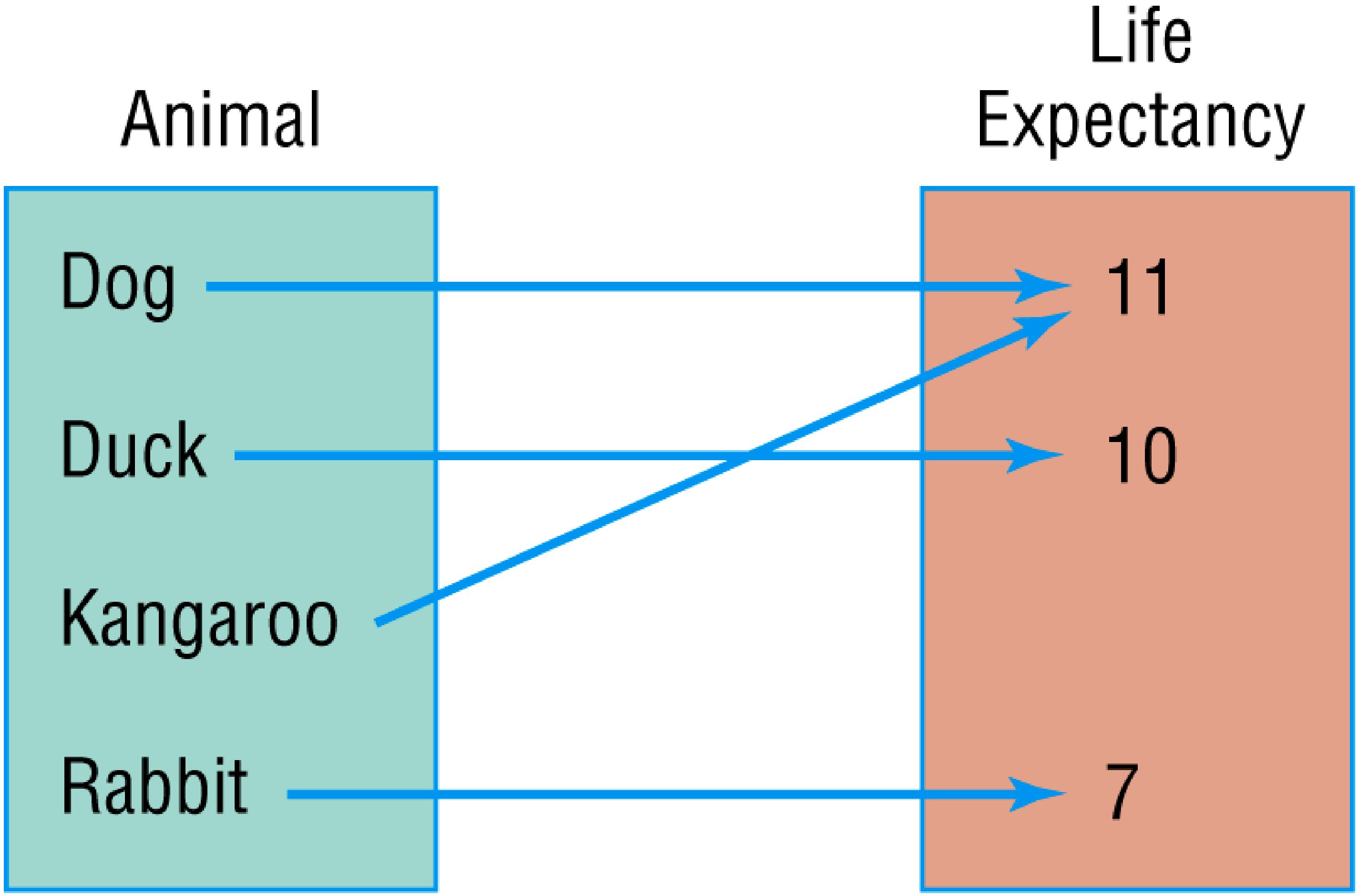


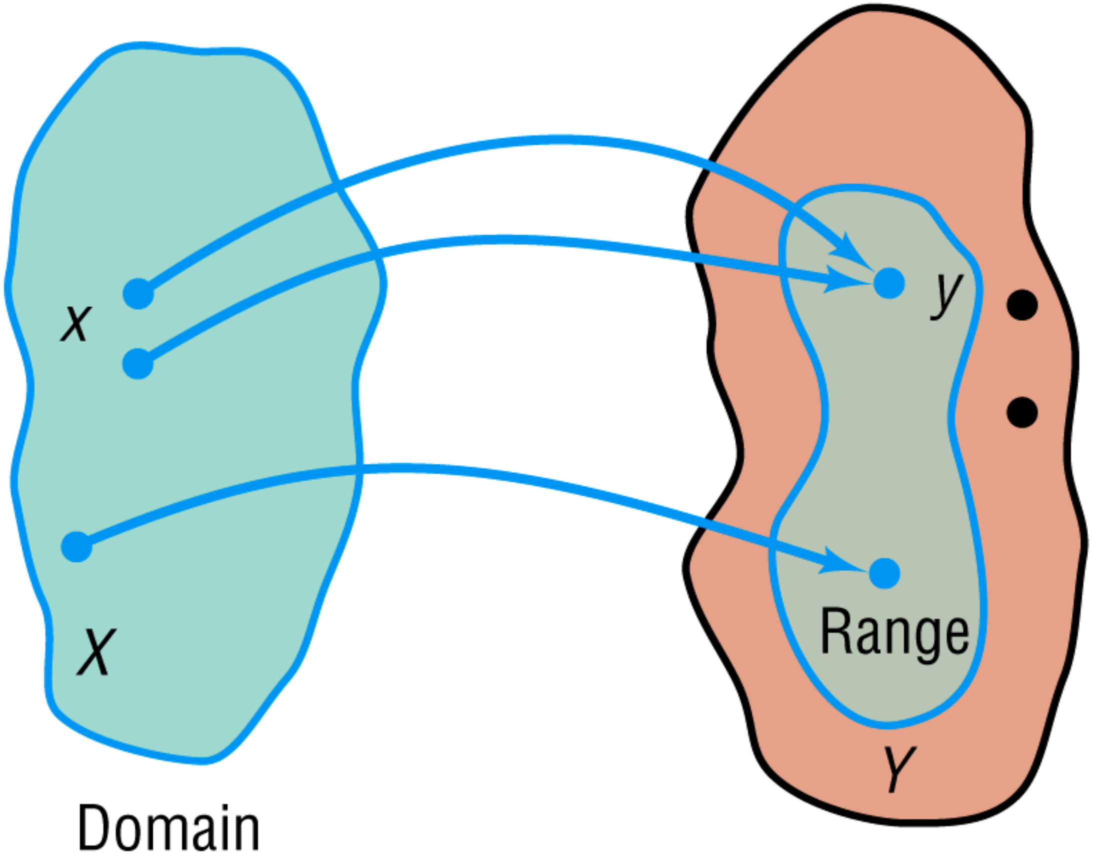
Figure: Animal Life Expectations



Definition

Let X and Y be two nonempty sets.* A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

Figure



Example

Determining Whether a Relation Is a Function

For each relation in Figures 6, 7, and 8, state the domain and range. Then determine whether the relation is a function.

(a) See Figure 6. For this relation, the input is the number of calories in a fast-food sandwich, and the output is the fat content (in grams).

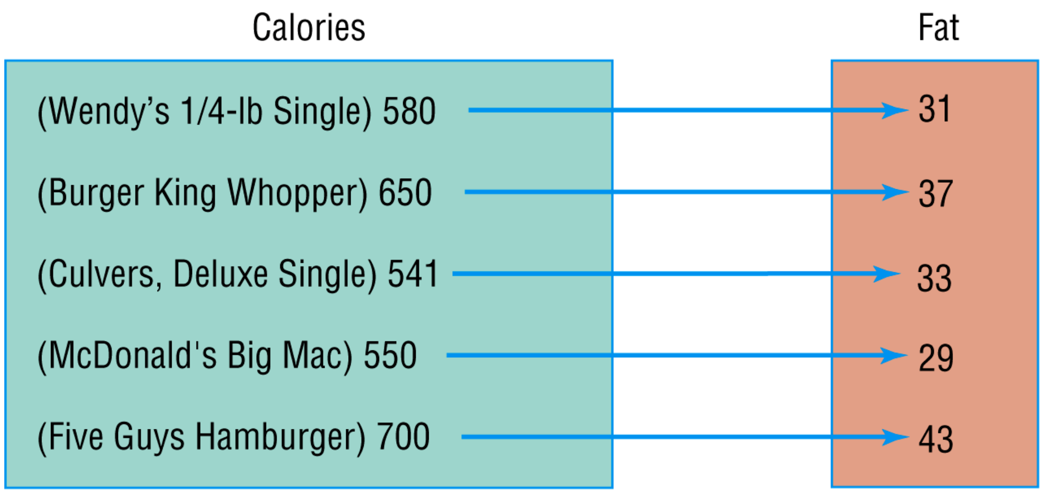


Figure 6

Source: Each company’s Web site

Example continued

(b) See Figure 7. For this relation, the inputs are gasoline stations in Harris County, Texas, and the outputs are the price per gallon of unleaded regular in March 2014.

(c) See Figure 8. For this relation, the inputs are the weight (in carats) of pear-cut diamonds and the outputs are the price (in dollars).

Figure 7

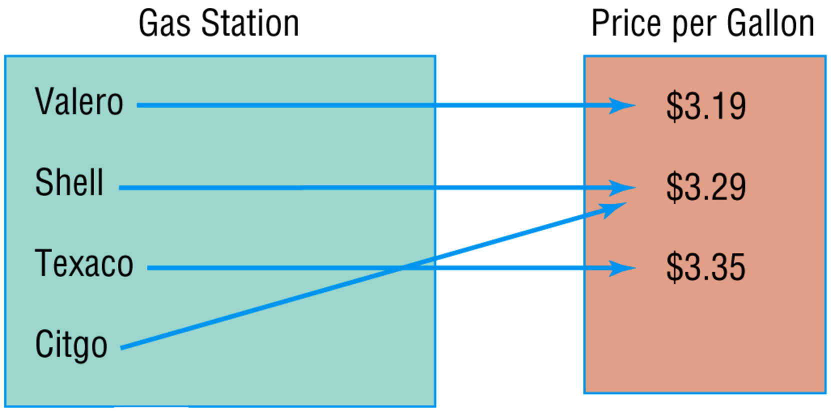


Figure 8

Source: Used with permission of Diamonds.com

Solution

(a) The domain of the relation is {541, 550, 580, 650, 700}, and the range of the relation is {29, 31, 33, 37, 43}. The relation in Figure 6 is a function because each element in the domain corresponds to exactly one element in the range.

(b) The domain of the relation is {Citgo, Shell, Texaco, Valero}. The range of the relation is {\$3.19, \$3.29, \$3.35}. The relation in Figure 7 is a function because each element in the domain corresponds to exactly one element in the range. Notice that it is okay for more than one element in the domain to correspond to the same element in the range (Shell and Citgo both sell gas for \$3.29 a gallon).

(c) The domain of the relation is {0.70, 0.71, 0.75, 0.78} and the range is {\$1529, \$1575, \$1765, \$1798, \$1952}. The relation in Figure 8 is not a function because not every element in the domain corresponds to exactly one element in the range. If a 0.71-carat diamond is chosen from the domain, a single price cannot be assigned to it.

Example

Determining Whether a Relation Is a Function

For each relation, state the domain and range. Then determine whether the relation is a function.

(a) $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$
(b) $\{(1, 4), (2, 4), (3, 5), (6, 10)\}$
(c) $\{(-3, 9), (-2, 4), (0, 0), (1, 1), (-3, 8)\}$

Solution

(a) The domain of this relation is {1, 2, 3, 4}, and its range is {4, 5, 6, 7}. This relation is a function because there are no ordered pairs with the same first element and different second elements.

(b) The domain of this relation is {1, 2, 3, 6}, and its range is {4, 5, 10}. This relation is a function because there are no ordered pairs with the same first element and different second elements.

(c) The domain of this relation is $\{-3, -2, 0, 1\}$, and its range is {0, 1, 4, 8, 9}. This relation is not a function because there are two ordered pairs, $(-3, 9)$ and $(-3, 8)$, that have the same first element and different second elements.

Example

Determining Whether an Equation Is a Function

Determine whether the equation $x^2 + y^2 = 1$ defines y as a function of x.

Solution

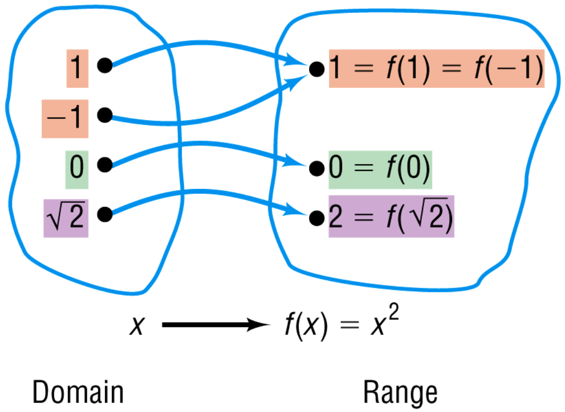
To determine whether the equation $x^2 + y^2 = 1$, which defines the unit circle, is a function, solve the equation for y.

$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1 - x^2}$$

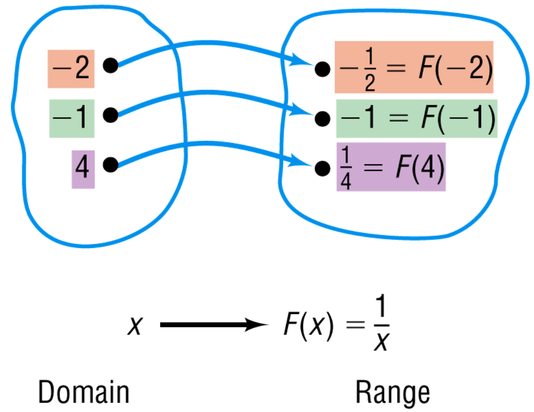
For values of x for which $-1 < x < 1$, two values of y result. For example, if $x = 0$, then $y = \pm 1$, so two different outputs result from the same input. This means that the equation $x^2 + y^2 = 1$ does not define a function.

Find the Value of a Function

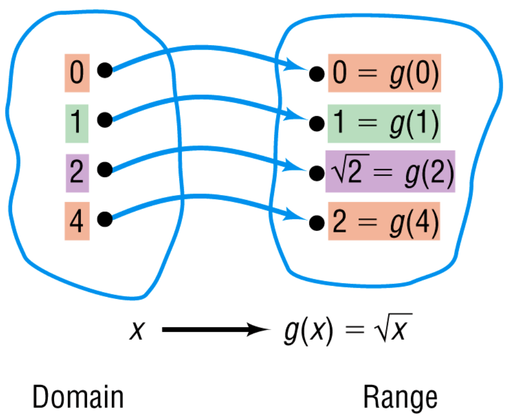
Some Functions



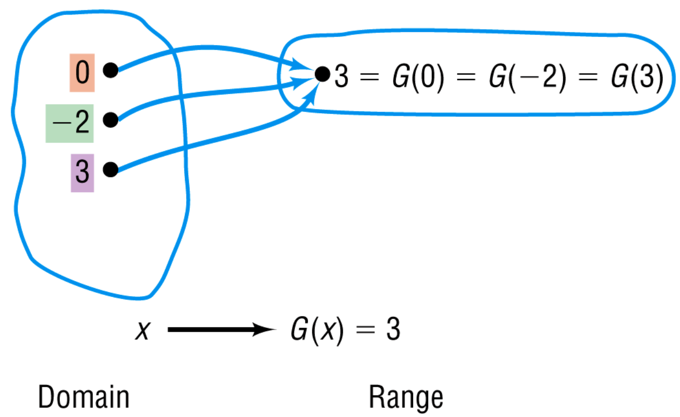
(a) $f(x) = x^2$



(b) $F(x) = \frac{1}{x}$

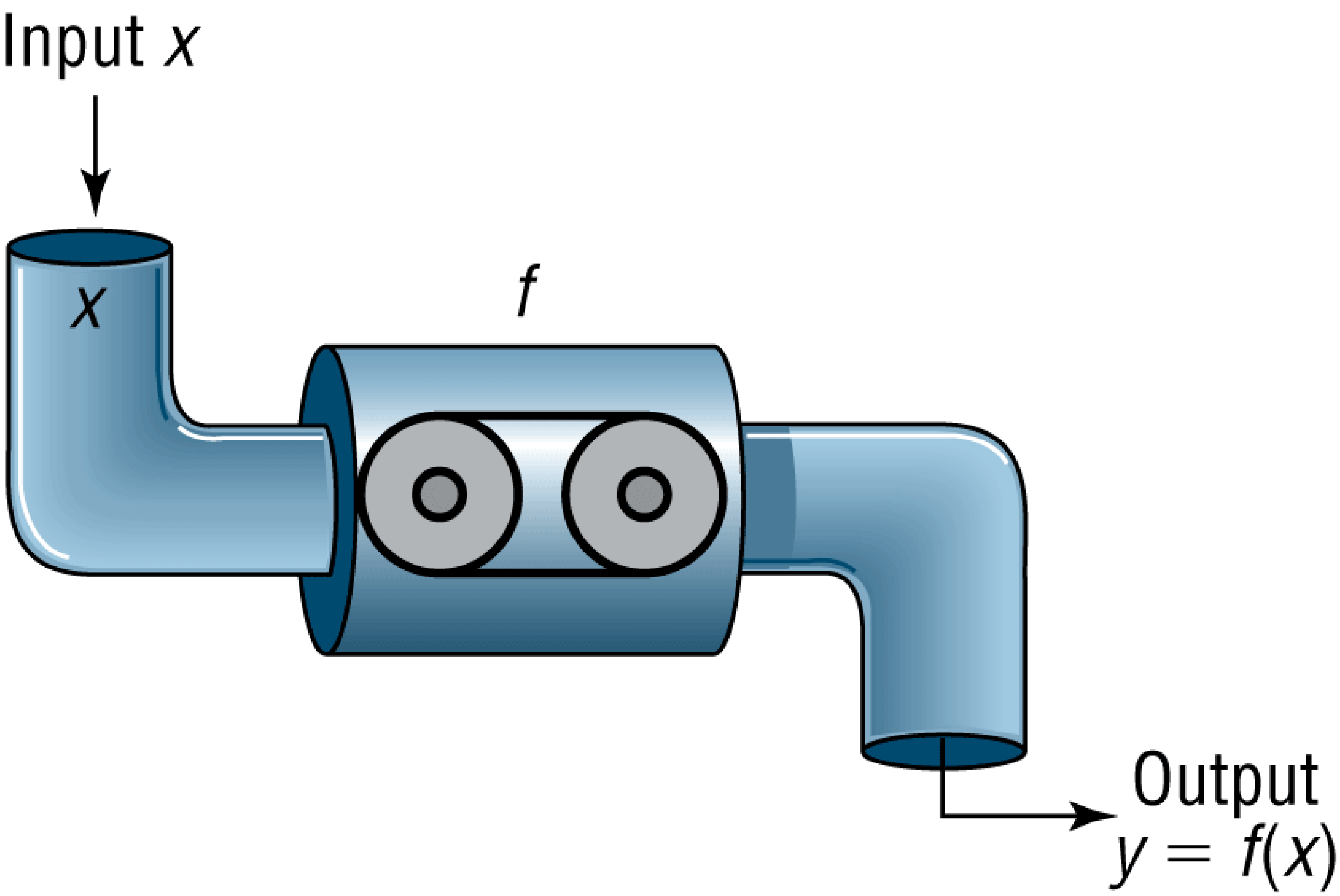


(c) $g(x) = \sqrt{x}$



(d) $G(x) = 3$

Figure: Input/Output Machine



Example

Finding Values of a Function

For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

(a) $f(3)$ (b) $f(x) + f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x + 3)$

Solution

(a) Substitute 3 for x in the equation for f, $f(x) = 2x^2 - 3x$, to get

$$f(3) = 2(3)^2 - 3(3) = 18 - 9 = 9$$

The image of 3 is 9.

(b) $f(x) + f(3) = (2x^2 - 3x) + (9) = 2x^2 - 3x + 9$

(c) Multiply the equation for f by 3.

$$3f(x) = 3(2x^2 - 3x) = 6x^2 - 9x$$

(d) Substitute $-x$ for x in the equation for f and simplify.

$$f(-x) = 2(-x)^2 - 3(-x) = 2x^2 + 3x$$ Notice the use of parentheses here.

Solution continued

(e) $-f(x) = -(2x^2 - 3x) = -2x^2 + 3x$

(f) Substitute $3x$ for x in the equation for f and simplify.

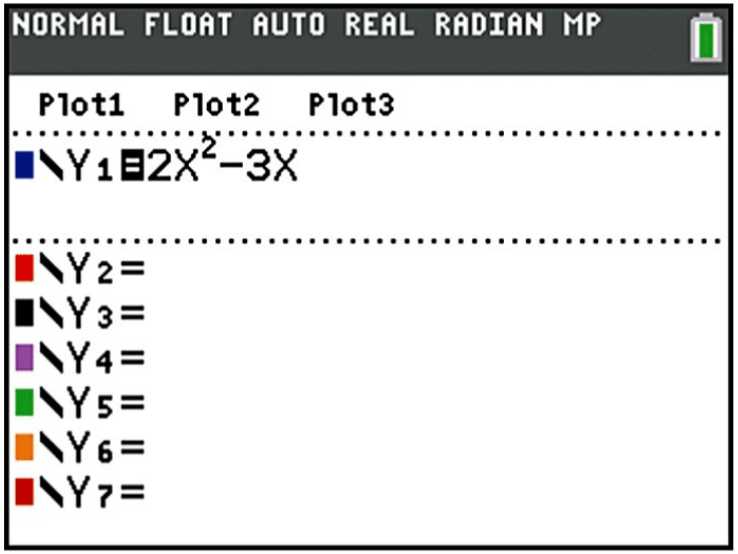
$$f(3x) = 2(3x)^2 - 3(3x) = 2(9x^2) - 9x = 18x^2 - 9x$$

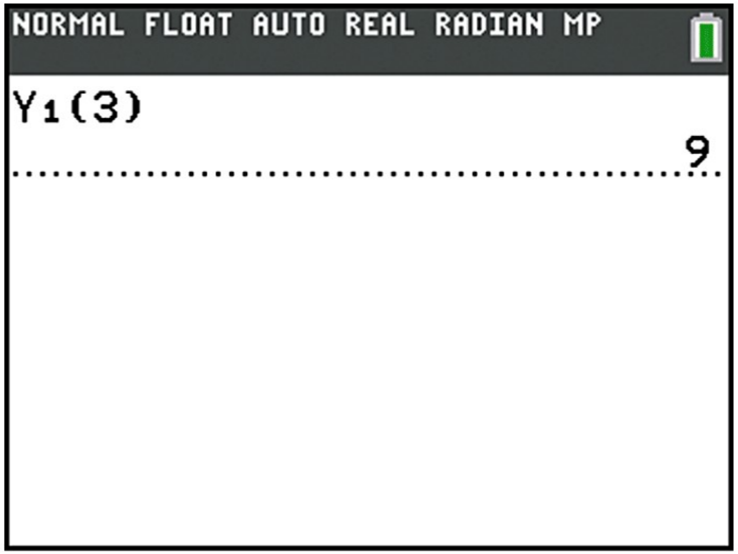
(g) Substitute $x + 3$ for x in the equation for f and simplify.

$$\begin{aligned} f(x+3) &= 2(x+3)^2 - 3(x+3) \\ &= 2(x^2 + 6x + 9) - 3x - 9 \\ &= 2x^2 + 12x + 18 - 3x - 9 \\ &= 2x^2 + 9x + 9 \end{aligned}$$

Figure

Evaluating $f(x) = 2x^2 - 3x$ for x = 3 on a TI-84 Plus C





Find the Difference Quotient of a Function

Definition

The **difference quotient** of a function f at x is given by

$$\frac{f(x+h)-f(x)}{h} \qquad h \neq 0 \qquad \textbf{(1)}$$

Example

Finding the Difference Quotient of a Function

Find the difference quotient of each function.

(a) $f(x) = 2x^2 - 3x$

(b) $f(x) = \frac{4}{x}$

(c) $f(x) = \sqrt{x}$

Solution

(a) $$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2 - 3(x+h)] - [2x^2 - 3x]}{h}$$

$\uparrow$ $f(x+h) = 2(x+h)^2 - 3(x+h)$

$$= \frac{2(x^2 + 2xh + h^2) - 3x - 3h - 2x^2 + 3x}{h}$$ Simplify.

$$= \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h}$$ Distribute and combine like terms.

$$= \frac{4xh + 2h^2 - 3h}{h}$$ Combine like terms.

$$= \frac{h(4x + 2h - 3)}{h}$$ Factor out h.

$$= 4x + 2h - 3$$ Divide out the h's.

Solution continued

(b) $$\frac{f(x+h)-f(x)}{h} = \frac{\frac{4}{x+h} - \frac{4}{x}}{h}$$ $f(x+h) = \frac{4}{x+h}$

$$= \frac{\frac{4x-4(x+h)}{x(x+h)}}{h}$$ Subtract.

$$= \frac{4x-4x-4h}{x(x+h)h}$$ Divide and distribute.

$$= \frac{-4h}{x(x+h)h}$$ Simplify.

$$= -\frac{4}{x(x+h)}$$ Divide out the factor h.

Solution continued

(c) $$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$$ $f(x+h)=\sqrt{x+h}$

$$=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$ Rationalize the numerator.

$$=\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $(A-B)(A+B)=A^2-B^2$

$$=\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2=x+h-x=h$

$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$ Divide out the factor h.

Find the Domain of a Function Defined by an Equation

Example

Finding the Domain of a Function

Find the domain of each of the following functions.

(a) $f(x) = x^2 + 5x$

(b) $g(x) = \dfrac{3x}{x^2 - 4}$

(c) $h(t) = \sqrt{4 - 3t}$

(d) $F(x) = \dfrac{\sqrt{3x + 12}}{x - 5}$

Solution

(a) The function says to square a number and then add five times the number. Since these operations can be performed on any real number, the domain of f is the set of all real numbers.

(b) The function g says to divide $3x$ by $x^2 - 4$. Since division by 0 is not defined, the denominator $x^2 - 4$ can never be 0, so x can never equal -2 or 2. The domain of the function g is $\{x \mid x \neq -2, x \neq 2\}$.

(c) The function h says to take the square root of $4 - 3t$. But only nonnegative numbers have real square roots, so the expression under the square root (the radicand) must be nonnegative (greater than or equal to zero). This requires that

$$\begin{aligned} 4 - 3t &\geq 0 \\ -3t &\geq -4 \\ t &\leq \frac{4}{3} \end{aligned}$$

The domain of h is $\left\{t \,\middle|\, t \leq \frac{4}{3}\right\}$, or the interval $\left(-\infty, \frac{4}{3}\right]$.

Solution continued

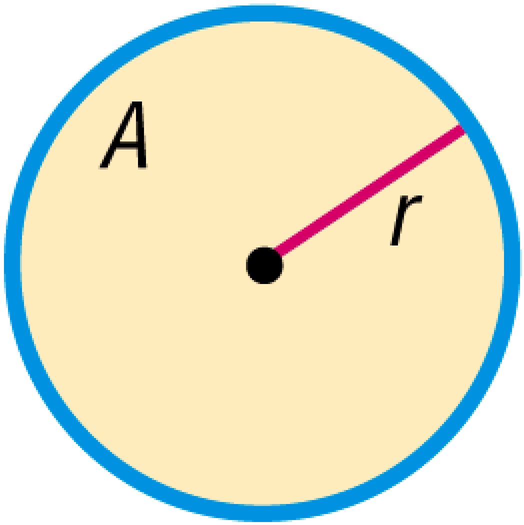
(d) The function F says to take the square root of $3x + 12$ and divide this result by $x - 5$. This requires that $3x + 12 \geq 0$, so $x \geq -4$, and also that $x - 5 \neq 0$, so $x \neq 5$. Combining these two restrictions, the domain of F is

$$\{x | x \geq -4, \quad x \neq 5\}.$$

Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical (the radicand) to be negative.

Figure: Circle of radius r



Form the Sum, Difference, Product, and Quotient of Two Functions

Definition

If f and g are functions:
The **sum** $\boldsymbol{f + g}$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Definition

The **difference** $f - g$ is the function defined by

$$(f - g)(x) = f(x) - g(x)$$

Definition

The **product** $\boldsymbol{f \cdot g}$ is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Definition

The **quotient** $\frac{f}{g}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

Example

Operations on Functions

Let f and g be two functions defined as

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{x}{x-1}$$

Find the following functions, and determine the domain in each case.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$

Solution

The domain of f is $\{x | x \neq -2\}$ and the domain of g is $\{x | x \neq 1\}$.

(a) $$(f+g)(x) = f(x) + g(x) = \frac{1}{x+2} + \frac{x}{x-1}$$

$$= \frac{x-1}{(x+2)(x-1)} + \frac{x(x+2)}{(x+2)(x-1)} = \frac{x^2+3x-1}{(x+2)(x-1)}$$

The domain of $f+g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f+g$ is $\{x | x \neq -2, x \neq 1\}$.

(b) $$(f-g)(x) = f(x) - g(x) = \frac{1}{x+2} - \frac{x}{x-1}$$

$$= \frac{x-1}{(x+2)(x-1)} - \frac{x(x+2)}{(x+2)(x-1)} = \frac{-(x^2+x+1)}{(x+2)(x-1)}$$

The domain of $f-g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f-g$ is $\{x | x \neq -2, x \neq 1\}$.

Solution continued

(c) $(f \cdot g)(x) = f(x) \cdot g(x) = \dfrac{1}{x+2} \cdot \dfrac{x}{x-1} = \dfrac{x}{(x+2)(x-1)}$

The domain of $f \cdot g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f \cdot g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{\dfrac{1}{x+2}}{\dfrac{x}{x-1}} = \dfrac{1}{x+2} \cdot \dfrac{x-1}{x} = \dfrac{x-1}{x(x+2)}$

The domain of $\dfrac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. Since $g(x) = 0$ when $x = 0$, we exclude 0 as well as -2 and 1 from the domain. The domain of $\dfrac{f}{g}$ is $\{x \mid x \neq -2, x \neq 0, x \neq 1\}$.