4 Exponential and Logarithmic Functions
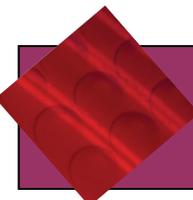
4.3 DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS
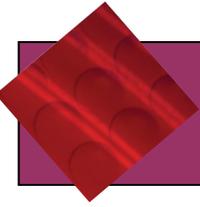
Introduction

Introduction

- In this section we differentiate these new functions and use their derivatives for graphing, optimization, and finding rates of change.

- We emphasize *natural* (base *e*) logs and exponentials, since most applications use these exclusively.

Derivatives of Logarithmic Functions

Derivatives of Logarithmic Functions

The rule for differentiating the natural logarithm function is as follows:

Derivative of ln x

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

The derivative of ln x is 1 over x

Example 1 – *DIFFERENTIATING A LOGARITHMIC FUNCTION*

Differentiate $f(x) = x^3 \ln x$.

Solution:

The function is a *product, x^3* times ln *x,* so we use the Product Rule.

$$\frac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^3 \frac{1}{x} = 3x^2 \ln x + x^2$$

From $x^3 \frac{1}{x} = x^2$

Derivative of the first   Second left alone   First left alone   Derivative of ln x

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Derivatives of Logarithmic Functions

The preceding rule, together with the Chain Rule, shows how to differentiate the natural logarithm of a *function.* For any differentiable function $f(x)$ that is positive:

Derivative of $\ln f(x)$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

The derivative of the natural log of a function is the derivative of the function over the function

Notice that the right-hand side does not involve logarithms at all.

Example 2 – *DIFFERENTIATING A LOGARITHMIC FUNCTION*

$$\frac{d}{dx}\ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

← Derivative of $x^2 + 1$

← Original function (without the ln)

Derivatives of Exponential Functions

Derivatives of Exponential Functions

The rule for differentiating the exponential function e^x is as follows:

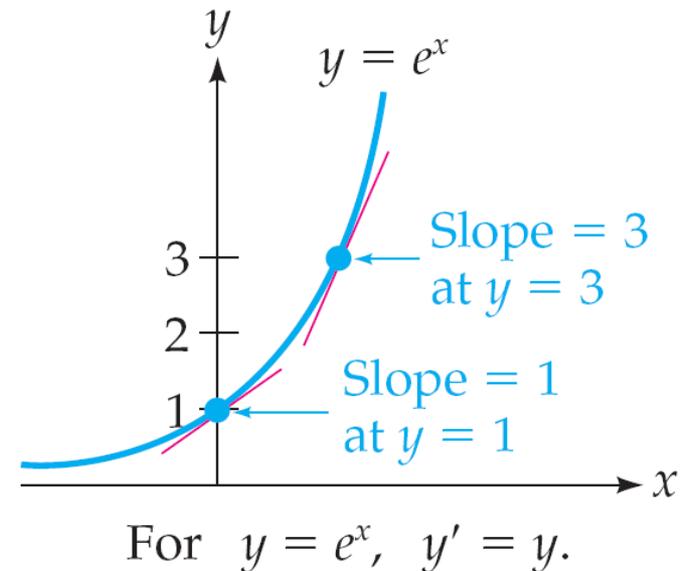**Derivative of e^x**

$$\frac{d}{dx} e^x = e^x$$

The derivative of e^x is simply e^x

The function e^x is unchanged by the operation of differentiation.

Derivatives of Exponential Functions

This rule can be interpreted graphically:

if $y = e^x$, then $y' = e^x$, so that $y = y'$.

This means that on the graph of $y = e^x$, the slope y' always equals the y-coordinate, as shown in the graph on the right.

$y = e^x$

Slope = 3 at $y = 3$

Slope = 1 at $y = 1$

For $y = e^x$, $y' = y$.

Since y' and y'' both equal e^x, they are always positive and the graph is always increasing and concave upwards.

Example 4 – *FINDING A DERIVATIVE INVOLVING e^x*

Find $\dfrac{d}{dx}\left(\dfrac{e^x}{x}\right).$

Solution:

Since the function is a quotient, we use the Quotient Rule:

Derivative of e^x

Derivative of x

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{x \cdot e^x - 1 \cdot e^x}{x^2} = \frac{xe^x - e^x}{x^2}$$

Derivatives of Exponential Functions

The rule for differentiating e^x, together with the Chain Rule, shows how to differentiate $e^{f(x)}$.

For any differentiable function $f(x)$:

Derivative of $e^{f(x)}$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$$

The derivative of e to a function is e to the function times the derivative of the function

That is, to differentiate $e^{f(x)}$ we simply "copy" the original $e^{f(x)}$ and then by the derivative of the exponent.

Example 6 – *DIFFERENTIATING AN EXPONENTIAL FUNCTION*

$$\frac{d}{dx} e^{x^4+1} = \underbrace{e^{x^4+1}}_{\text{Copied}} (4x^3)$$

Reversing the order

Derivative of the exponent

$$= 4x^3 e^{x^4+1}$$

Derivatives of Exponential Functions

The formulas for differentiating natural logarithmic and exponential functions are summarized as follows, with *f*(*x*) written simply as *f*.

Logarithmic Formulas	Exponential Formulas	
$\dfrac{d}{dx}\ln x = \dfrac{1}{x}$	$\dfrac{d}{dx}e^x = e^x$	Top formulas apply only to $\ln x$ and e^x
$\dfrac{d}{dx}\ln f = \dfrac{f'}{f}$	$\dfrac{d}{dx}e^f = e^f \cdot f'$	Bottom formulas apply to ln and e of a *function*

Example 8 – *DIFFERENTIATING A LOGARITHMIC AND EXPONENTIAL FUNCTION*

Find the derivative of ln (1 + e^x).

Solution:

$$\frac{d}{dx} \ln(1 + e^x) \;=\; \frac{\dfrac{d}{dx}(1 + e^x)}{1 + e^x} \;=\; \frac{e^x}{1 + e^x}$$

Using $\dfrac{d}{dx} \ln f = \dfrac{f'}{f}$ Working out the numerator

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Derivatives of Exponential Functions

Derivative of e^{kx}

$$\frac{d}{dx} e^{kx} = ke^{kx}$$

For any constant k

This formula says that the rate of change (the derivative) of e^{kx} is proportional to itself. That is, the function satisfies the *differential equation*

$$y' = ky$$

We noted this earlier when we observed that in exponential growth a quantity *grows in proportion to itself*.

Derivatives of Exponential Functions

These differentiation formulas enable us to find instantaneous rates of change of logarithmic and exponential functions.

In many applications the variable stands for time, so we use *t* instead of *x*.

Example 9 – *FINDING A RATE OF IMPROVEMENT OF A SKILL*

After t weeks of practice a pole vaulter can vault

$$H(t) = 15 - 11e^{-0.1t}$$

feet. Find the rate of change of the athlete's jumps after

a. 0 weeks (at the beginning of training)

b. 12 weeks

Solution:

We differentiate to find the rate of change

$$H'(t) = \underbrace{-11(-0.1)e^{-0.1t}}_{\text{Using } \frac{d}{dt} e^{kt} = ke^{kt}} = \underbrace{1.1e^{-0.1t}}_{\text{Simplifying}}$$

Differentiating $15 - 11e^{-0.1t}$

Example 9 – *Solution*

cont'd

a. For the rate of change after 0 weeks:

$$H'(0) = 1.1e^{-0.1(0)} = 1.1e^0 = 1.1 \qquad H'(t) = 1.1e^{-0.1t} \text{ with } t = 0$$

b. After 12 weeks:

$$H'(12) = 1.1e^{-0.1(12)}$$

$$= 1.1e^{-1.2} \approx 1.1(0.30) \qquad H'(t) = 1.1e^{-0.1t} \text{ with } t = 12$$

$$= 0.33 \qquad \text{Using a calculator}$$

At first, the vaults increased by 1.1 feet per week. After 12 weeks, the gain was only 0.33 foot (about 4 inches) per week.

Derivatives of Exponential Functions

This result is typical of learning a new skill: early improvement is rapid, later improvement is slower.

This trend is called *diminishing returns,* and may be seen in the leveling off of the polevault heights in the graph below.

$$H(t) = 15 - 11e^{-0.1t}$$
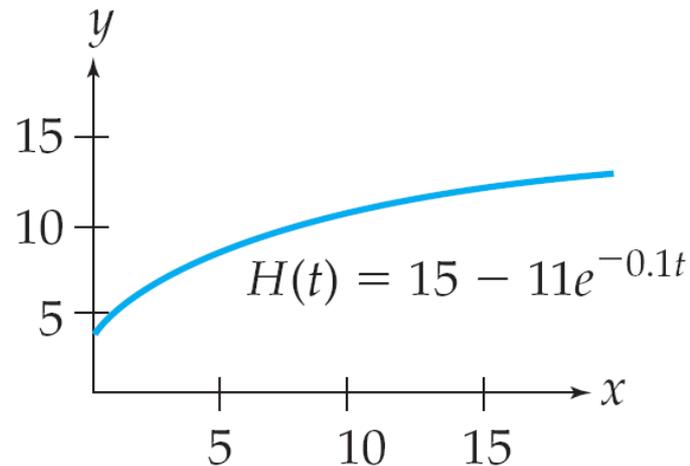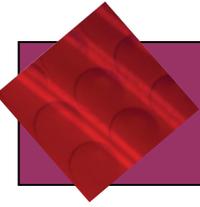
Maximizing Consumer Expenditure

Maximizing Consumer Expenditure

The amount of a commodity that consumers will buy depends on the price of the commodity.

For a commodity whose price is p, let the consumer demand be given by a function $D(p)$. Multiplying the number of units $D(p)$ by the price p gives the total *consumer expenditure* for the commodity.

Consumer Demand and Expenditure

Let $D(p)$ be the consumer demand at price p. Then the consumer expenditure is

$$E(p) = p \cdot D(p)$$

Example 10 – *MAXIMIZING CONSUMER EXPENDITURE*

If consumer demand for a commodity is $D(p) = 10{,}000e^{-0.02p}$ units per week, where p is the selling price, find the price that maximizes consumer expenditure.

Solution:

Using the preceding formula for consumer expenditure,

$$E(p) = p \cdot 10{,}000\, e^{-0.02p}$$

E(p) = p · D(p)

$$= 10{,}000pe^{-0.02p}$$

To maximize $E(p)$ we differentiate:

Using the Product Rule to differentiate $E(p) = 10{,}000p \cdot e^{-0.02p}$

$E'(p) =$ Derivative of $10{,}000p$ $\quad e^{-0.02p} + 10{,}000p$ Derivative of $e^{-0.02p}$

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Example 10 – *Solution*

cont'd

$= 10{,}000e^{-0.02p} - 200pe^{-0.02p}$ Simplifying

$= 200e^{-0.02p}(50 - p)$ Factoring

CN: $p = 50$ Critical number from $(50 - p)$ (since e to a power is never zero)

We calculate E'' for the second-derivative test:

$E''(p) = 200(-0.02)e^{-0.02p}(50 - p) + 200e^{-0.02p}(-1)$

From $E'(p) = 200e^{-0.02p} \cdot (50 - p)$ using the Product Rule

$= -4e^{-0.02p}(50 - p) - 200e^{-0.02p}$ Simplifying
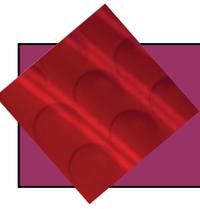
Example 10 – *Solution*

cont'd

At the critical number *p* = 50,

$$E''(50) = -4e^{-0.02(50)}(50 - 50) - 200e^{-0.02(50)}$$ Substituting *p* = 50

$$= \frac{-200}{e} \cdot {}^{-1}$$ Simplifying

$$=$$

E″ is negative, so the expenditure *E*(*p*) is maximized at *p* = 50:

Consumer expenditure is maximized at price $50.

Graphing Logarithmic and Exponential Functions

Graphing Logarithmic and Exponential Functions

To graph logarithmic and exponential functions using a graphing calculator, we first find critical points and possible inflection points, and then graph the function on a window including these points.

(If graphing "by hand," we would make sign diagrams for the first and second derivatives and then sketch the graph).

Example 11 – *GRAPHING AN EXPONENTIAL FUNCTION*

Graph $f(x) = e^{-x^2/2}$.

Solution:

As before, we write the function as $f(x) = e^{-\frac{1}{2}x^2}$.

The derivative is

$$f'(x) = e^{-\frac{1}{2}x^2}(-x) = -xe^{-\frac{1}{2}x^2} \qquad \text{Using} \quad \frac{d}{dx}e^f = e^f \cdot f'$$

↑ — Derivative of the exponent

CN: $x = 0$ Critical number is 0

$y = 1$ From $y = e^{-\frac{1}{2}x^2}$ evaluated at $x = 0$

The second derivative is

$$f''(x) = (-1)e^{-\frac{1}{2}x^2} - xe^{-\frac{1}{2}x^2}(-x)$$

From $f'(x) = -x \cdot e^{-\frac{1}{2}x^2}$ using the Product Rule

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Example 11 – *Solution*

cont'd

$$= -e^{-\frac{1}{2}x^2} + x^2 e^{-\frac{1}{2}x^2} \qquad \text{Simplifying}$$

$$= e^{-\frac{1}{2}x^2}(-1 + x^2) \qquad \text{Factoring}$$

$$= (x^2 - 1)e^{-\frac{1}{2}x^2} \qquad \text{Rearranging}$$

$$= (x + 1)(x - 1)e^{-\frac{1}{2}x^2} \qquad \text{Factoring}$$

$$x = \pm 1 \qquad \text{Where } f'' = 0$$

$$y = e^{-\frac{1}{2}} \approx 0.6 \qquad \text{From } \quad y = e^{-\frac{1}{2}x^2}$$
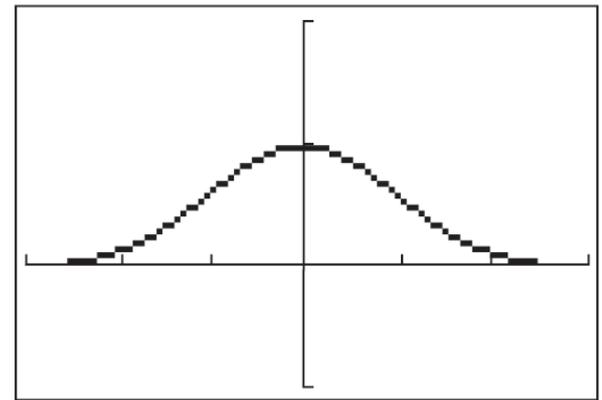$$\text{evaluated at} \quad x = \pm 1$$

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Graphing Logarithmic and Exponential Functions

Based on these values, we choose the graphing window as follows.

For the *x*-values we choose [–3, 3] (to include 0 and ±1 and beyond), and for the *y*-values we choose [–1, 2] (to include 1 and 0.6 and above and below).

$$f(x) = e^{-x^2/2} \text{ on } [-3, 3] \text{ by } [-1, 2]$$

This window gives the graph on the right.

(Many other windows would be just as good, and after seeing the graph you might want to adjust the window.)