4 Exponential and Logarithmic Functions

James King-Holmes/Photo Researchers, Inc.

4.1 EXPONENTIAL FUNCTIONS

Introduction

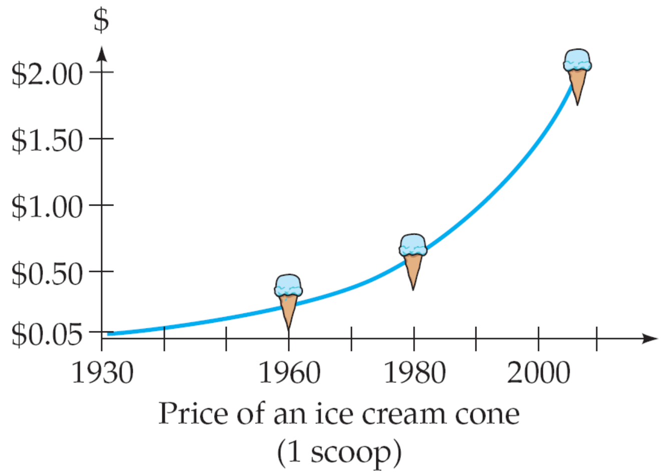
Introduction

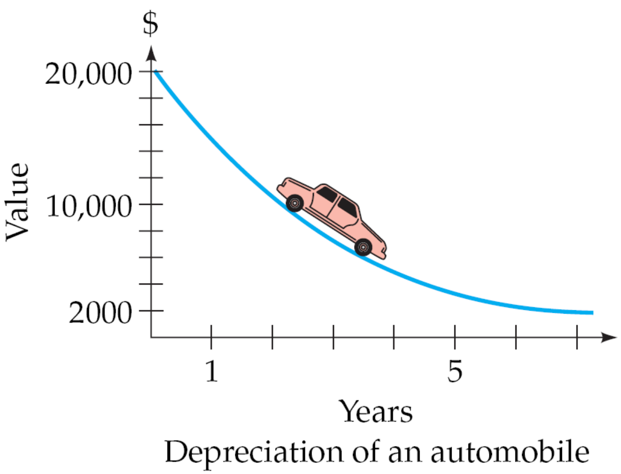
- We introduced **exponential** and **logarithmic functions,** two of the most useful functions in all of mathematics.

- In this section we develop their properties and apply them to a wide variety of problems.

Introduction

- We begin with exponential functions, showing how they are used to model the processes of growth and decay.





- We will also define the very important mathematical constant ***e*.**

Exponential Functions

Exponential Functions

A function that has a variable in an exponent, such as $f(x) = 2^x$, is called an *exponential function.* The number being raised to the power is called the **base.**

$$f(x) = \underbrace{2}_{\text{Base}}{}^{\overbrace{x}^{\text{Exponent}}}$$

More formally:

Exponential Functions

For any number $a > 0$, the function

$$f(x) = a^x$$

is called an *exponential function* with base a and exponent (or power) x.

Brief Examples

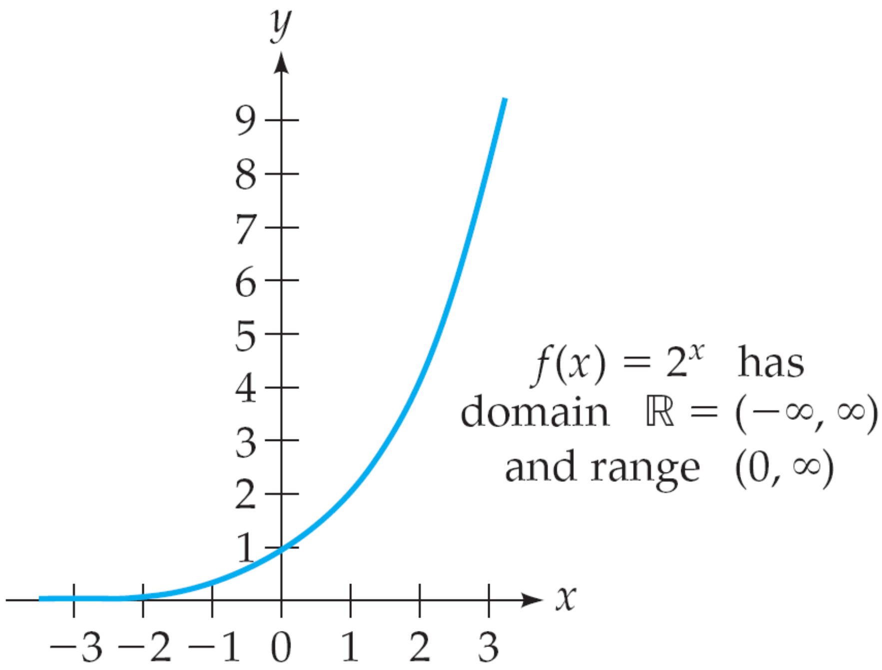
$f(x) = 2^x$ has base 2

$f(x) = \left(\frac{1}{2}\right)^x$ has base $\frac{1}{2}$

Exponential Functions

The table below shows some values of the exponential function $f(x) = 2^x$, and its graph (based on these points) is shown on the right.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



Graph of $y = 2^x$

Exponential Functions

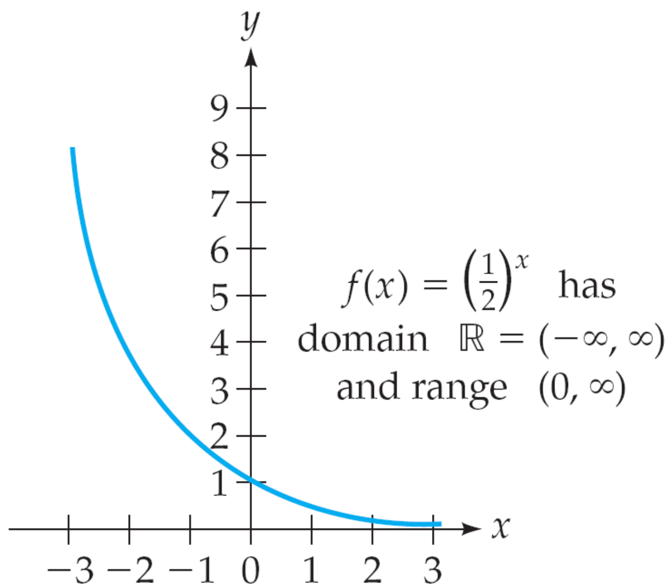
Clearly, the exponential function 2^x is quite different from the parabola x^2.

In particular, it is not symmetric about the *y*-axis and has the *x*-axis as a horizontal asymptote $(\lim_{x \to -\infty} 2^x = 0)$.

Exponential Functions

The following table shows some values of the exponential function $f(x) = \left(\frac{1}{2}\right)^x$ and its graph is shown to the right of the table. Notice that it is the mirror image of the curve $y = 2^x$, with both passing through the point (0, 1).

x	$y = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$



Graph of $y = \left(\frac{1}{2}\right)^x$

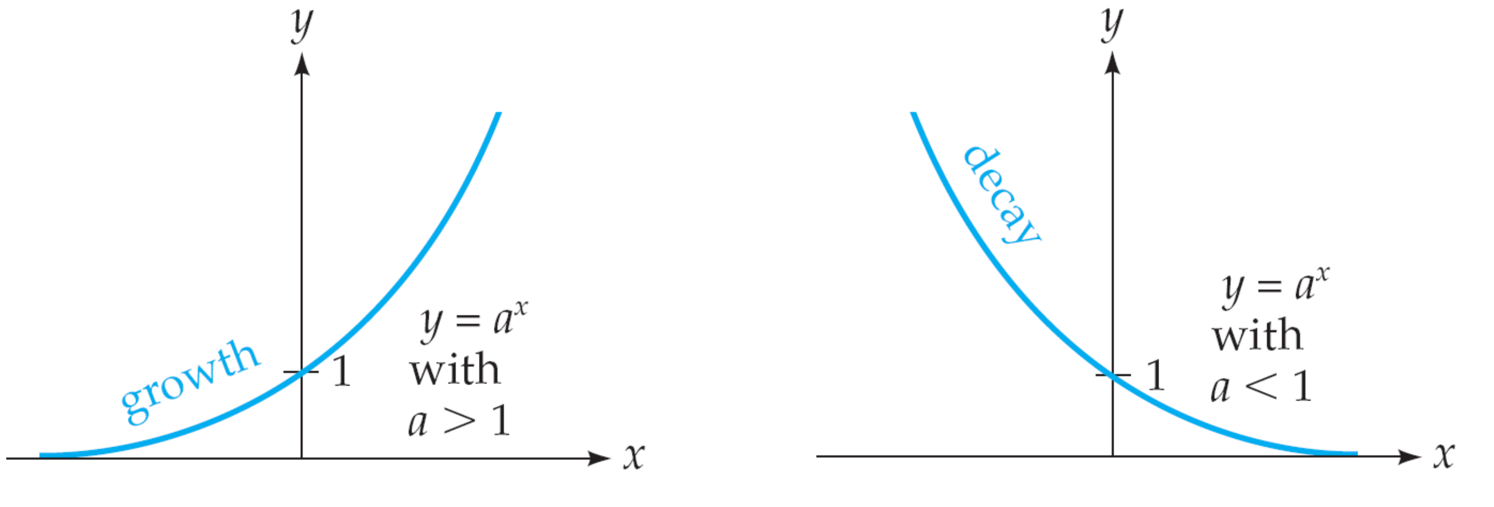
Exponential Functions

We can define an exponential function $f(x) = a^x$ for any positive base *a.*

We always take the base to be positive, so for the rest of this section *the letter a will stand for a positive constant.*

Exponential Functions

Exponential functions with bases $a > 1$ are used to model *growth,* as in populations or savings accounts, and exponential functions with bases $a < 1$ are used to model *decay,* as in depreciation. (For base $a = 1$, the graph is a horizontal line, since $1^x = 1$ for all x.)



$y = a^x$ slopes *upward* for base $a > 1$.

$y = a^x$ slopes *downward* for $a < 1$.

Compound Interest

Compound Interest

Money invested at compound interest grows exponentially.

Banks always state *annual* interest rates, but the compounding may be done more frequently.

For example, if a bank offers 8% compounded quarterly, then each quarter you get 2% (one quarter of the annual 8%), so that 2% of your money is added to the account each quarter.

If you begin with *P* dollars (the **principal**), at the end of the first quarter you would have *P* dollars plus 2% of *P* dollars:

$$\begin{pmatrix}\text{Value after}\\ \text{1 quarter}\end{pmatrix} = P + 0.02P = P \cdot (1 + 0.02) \qquad \text{Factoring out the } P$$

Compound Interest

Notice that increasing a quantity by 2% is the same as multiplying it by (1 + 0.02).

Since a year has 4 quarters, *t* years will have 4*t* quarters.

Therefore, to find the value of your account after *t* years, we simply multiply the principal by (1 + 0.02) a total of 4*t* times, obtaining:

$$\begin{pmatrix}\text{Value after}\\ t\text{ years}\end{pmatrix} = P\cdot\overbrace{(1+0.02)\cdot(1+0.02)\cdots(1+0.02)}^{4t\text{ times}}$$

$$= P\cdot(1+0.02)^{4t}$$

Compound Interest

The 8%, which gave the $\frac{0.08}{4} = 0.02$ quarterly rate, can be replaced by any interest rate *r* (written in decimal form), and the 4 can be replaced by any number *m* of compounding periods per year.

Compound Interest

For P dollars invested at annual interest rate r compounded m times a year for t years,

$$\begin{pmatrix}\text{Value after} \\ t \text{ years}\end{pmatrix} = P \cdot \left(1 + \frac{r}{m}\right)^{mt}$$

r = annual rate
m = periods per year
t = number of years

For example, for monthly compounding we would use $m = 12$ and for daily compounding $m = 365$ (the number of days in the year).

Example 1 – *FINDING A VALUE UNDER COMPOUND INTEREST*

Find the value of $4000 invested for 2 years at 12%

compounded quarterly.

Solution $$4000 \cdot \left(1 + \underbrace{\frac{0.12}{4}}_{0.03}\right)^{4 \cdot 2} = 4000(1 + 0.03)^8$$

$P \cdot \left(1 + \frac{r}{m}\right)^{mt}$
with $P = 4000$,
$r = 0.12$, $m = 4$,
and $t = 2$

$= 4000 \cdot 1.03^8$

≈ 5067.08 Using a calculator

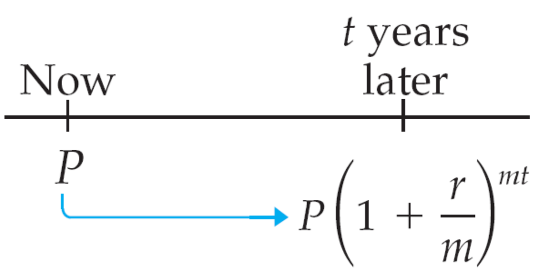
The value after 2 years will be $5067.08.

Present Value

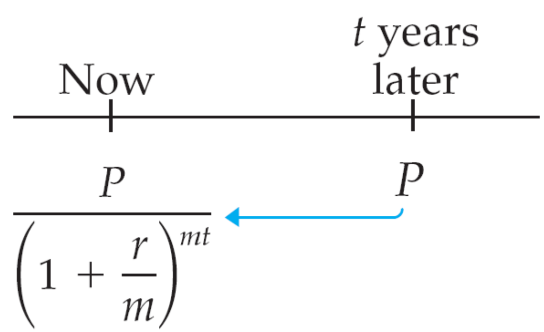
Present Value

The value to which a sum will grow under compound interest is often called its *future value.*

Reversing the order, we can speak of the *present value* of a future payment.



Future value



Present value

To find the *future* value we *multiply* by $(1 + r/m)^{mt}$, and so to find the *present* value, we *divide* by $(1 + r/m)^{mt}$.

Present Value

Present Value

For a future payment of P dollars at annual interest rate r compounded m times a year to be paid in t years,

$$\begin{pmatrix}\text{Present}\\ \text{value}\end{pmatrix} = \frac{P}{\left(1+\dfrac{r}{m}\right)^{mt}}$$

r = annual rate
m = periods per year
t = number of years

Example 2 – *FINDING PRESENT VALUE*

Find the present value of $5000 to be paid 8 years from now at 10% interest compounded semiannually.

Solution:

For semiannual compounding (m = 2), the formula gives

$$\frac{P}{\left(1+\frac{r}{m}\right)^{mt}} = \frac{5000}{\left(1+\frac{0.10}{2}\right)^{2\cdot 8}} \qquad P = 5000, \quad r = 0.10, \quad m = 2, \quad \text{and} \quad t = 8$$

$$= \frac{5000}{(1+0.05)^{16}} \approx 2290.56 \qquad \text{Using a calculator}$$

Therefore, the present value of the $5000 is just $2290.56.

Depreciation by a Fixed Percentage

Depreciation by a Fixed Percentage

Depreciation by a fixed percentage means that a piece of equipment loses the same percentage of its current value each year.

Losing a percentage of value is like compound interest but with a *negative* interest rate.

Therefore, we use the compound interest formula $P(1 + r/m)^{mt}$ with $m = 1$ (since depreciation is annual) and with r being negative.

Example 3 – *DEPRECIATING AN ASSET*

A car worth $15,000 depreciates in value by 40% each year. How much is it worth after 3 years?

Solution:

The car loses 40% of its value each year, which is equivalent to an interest rate of *negative* 40%.

The compound interest formula gives

$$15{,}000(1 - 0.40)^3 = 15{,}000(0.60)^3 \qquad P(1 + r/m)^{mt} \text{ with } P = 15{,}000,\ r = -0.40,\ m = 1, \text{ and } t = 3$$

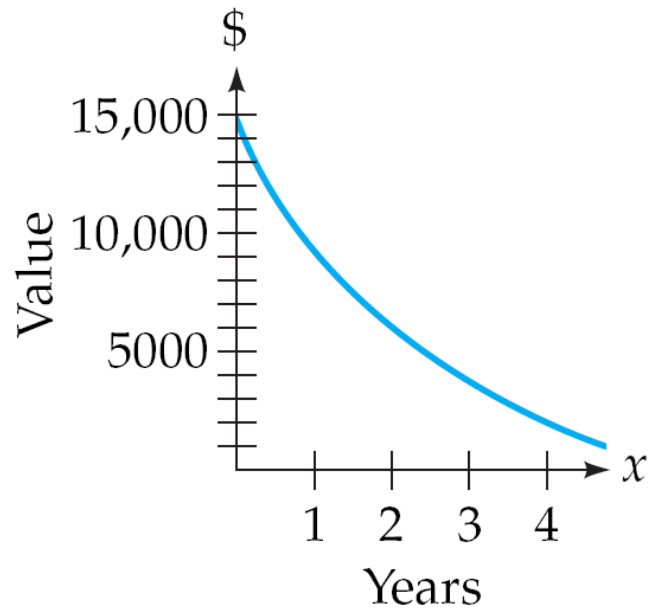
$$= \underbrace{\$3240}_{\text{Using a calculator}}$$

Example 3 – *Solution*

cont'd

The exponential function $f(x) = 15{,}000(0.60)^x$, giving the value of the car after x years of depreciation, is graphed below.

Notice that a yearly loss of 40% means that 60% of the value is retained each year.

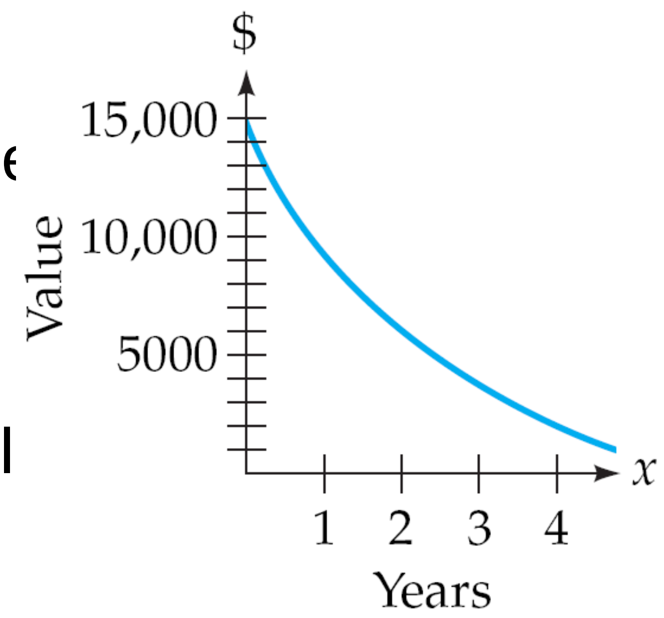


Depreciation by a Fixed Percentage

The graph shows that depreciation by a fixed percentage is quite different from "straight-line" depreciation.

Under straight-line depreciation the same *dollar* value is l

under fixed-percentage depreciation the same *percentage* of value is lost each year, resulting in larger dollar losses in the early years and smaller dollar losses in later years.



Depreciation by a Fixed Percentage

Depreciation by a fixed percentage (also called the **declining balance method**) is one type of **accelerated depreciation.**

The method of depreciation that one uses depends on how one chooses to estimate value, and in practice is often determined by the tax laws.

The Number *e*

The Number *e*

Imagine that a bank offers 100% interest, and that you deposit $1 for 1 year.

Let us see how the value changes under different types of compounding.

For *annual* compounding, your $1 would in a year grow to $2 (the original dollar plus a dollar interest).

The Number *e*

For *quarterly* compounding, we use the compound interest formula with $P = 1$, $r = 1$ (for 100%), $m = 4$, and $t = 1$:

$$1\left(1 + \frac{1}{4}\right)^{4\cdot 1} = 1(1 + 0.25)^4 = (1.25)^4 \approx 2.44 \qquad P\left(1 + \frac{r}{m}\right)^{mt}$$

or \$2.44, an improvement of 44 cents over annual compounding.

For *daily* compounding, the value after a year would be

$$\left(1 + \frac{1}{365}\right)^{365} \approx 2.71$$

$$m = 365 \text{ periods}$$

$$\frac{r}{m} = \frac{100\%}{365} = \frac{1}{365}$$

an increase of 27 cents over quarterly compounding.

The Number *e*

Clearly, if the interest rate, the principal, and the amount of time stay the same, the value increases as the compounding is done more frequently.

In general, if the compounding is done *m* times a year, the value of the dollar after a year will be

$$\left(\begin{array}{c}\text{Value of \$1 after 1 year at 100\%} \\ \text{interest compounded } m \text{ times a year}\end{array}\right) = \left(1 + \frac{1}{m}\right)^m$$

The Number *e*

The following table shows the value of $\left(1+\frac{1}{m}\right)^m$ for various values of *m*.

Value of \$1 at 100% Interest Compounded *m* Times a Year for 1 Year

m	$\left(1+\frac{1}{m}\right)^m$	Answer (rounded)	
1	$\left(1+\frac{1}{1}\right)^1$	$= 2.00000$	Annual compounding
4	$\left(1+\frac{1}{4}\right)^4$	≈ 2.44141	Quarterly compounding
365	$\left(1+\frac{1}{365}\right)^{365}$	≈ 2.71457	Daily compounding
10,000	$\left(1+\frac{1}{10{,}000}\right)^{10{,}000}$	≈ 2.71815	
100,000	$\left(1+\frac{1}{100{,}000}\right)^{100{,}000}$	≈ 2.71827	
1,000,000	$\left(1+\frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$	≈ 2.71828	Answers agree to five decimal places
10,000,000	$\left(1+\frac{1}{10{,}000{,}000}\right)^{10{,}000{,}000}$	≈ 2.71828	

The Number *e*

Notice that as m increases, the values in the right-hand column seem to be settling down to a definite value, approximately 2.71828.

That is, as m approaches infinity, the limit of $\left(1+\frac{1}{m}\right)^m$ is approximately 2.71828.

This particular number is very important in mathematics, and is given the name e (just as 3.14159…. is given the name π).

The Number *e*

In the following definition we use the letter *n* to state the definition in its traditional form.

The Constant *e*

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\ldots$$

The dots mean that the decimal expansion goes on forever

The same *e* appears in probability and statistics in the formula for the “bell-shaped” or “normal” curve. Its value has been calculated to several million decimal places, and its value to 15 decimal places is $e \approx 2.718281828459045$.

Continuous Compounding of Interest

Continuous Compounding of Interest

This kind of compound interest, the limit as the compounding frequency approaches infinity, is called **continuous compounding.**

We have shown that \$1 at 100% interest compounded continuously for 1 year would be worth precisely *e* dollars (about \$2.72).

Continuous Compounding of Interest

The formula for continuous compound interest at other rates is as follows.

Continuous Compounding

For P dollars invested at annual interest rate r compounded continuously for t years,

$$\begin{pmatrix}\text{Value after} \\ t \text{ years}\end{pmatrix} = Pe^{rt}$$

Example 4 – *FINDING VALUE WITH CONTINUOUS COMPOUNDING*

Find the value of \$1000 at 8% interest compounded continuously for 20 years.

Solution:

We use the formula Pe^{rt} with $P = 1000$, $r = 0.08$, and $t = 20$

$$Pe^{rt} = \underbrace{1000}_{P} \cdot e^{\overset{\uparrow}{\underset{r}{(0.08)}}\overset{\uparrow}{\underset{n}{(20)}}}$$

$$= 1000 \cdot \underbrace{e^{1.6}}_{4.95303} \approx \$4953.03$$

$e^{1.6}$ is usually found using the $\boxed{\text{2nd}}$ and $\boxed{\text{LN}}$ keys

Present Value with Continuous Compounding

Present Value with Continuous Compounding

As before, the value that a sum will attain in *t* years is often called its *future* value, and the current value of a future sum is its *present* value.

Under continuous compounding, to find future value we multiply *P* by e^{rt}, and so to find *present* value we *divide* by e^{rt}.

Present Value with Continuous Compounding

For a future payment of P dollars at annual interest rate r compounded continuously to be paid in t years,

$$\begin{pmatrix}\text{Present}\\ \text{value}\end{pmatrix} = \frac{P}{e^{rt}} = Pe^{-rt}$$

Example 5 – *FINDING PRESENT VALUE WITH CONTINUOUS COMPOUNDING*

The present value of \$5000 to be paid in 10 years, at 7% interest compounded continuously, is

$$\frac{5000}{e^{0.07\cdot 10}} = \frac{5000}{e^{0.7}} \approx \$2482.93 \qquad \text{Using a calculator}$$

Intuitive Meaning of Continuous Compounding

Intuitive Meaning of Continuous Compounding

Under quarterly compounding, your money is, in a sense, earning interest throughout the quarter, but the interest is not added to your account until the end of the quarter.

Under continuous compounding, the interest is added to your account *as it is earned,* with no delay.

The extra earnings in continuous compounding come from this “instant crediting” of interest, since then your interest starts earning more interest immediately.

How to Compare Interest Rates

How to Compare Interest Rates

How do you compare different interest rates, such as 16% compounded quarterly and 15.8% compounded continuously?

You simply see what each will do for a deposit of $1 for 1 year.

Example 6 – *COMPARING INTEREST RATES*

Which gives a better return: 16% compounded quarterly or 15.8% compounded continuously?

Solution:

For 16% compounded quarterly (on $1 for 1 year),

$$1 \cdot \left(1 + \frac{0.16}{4}\right)^4 = (1 + 0.04)^4 \approx 1.170 \qquad \text{Using } P\left(1 + \frac{r}{m}\right)^{mt}$$

For 15.8% compounded continuously,

$$1 \cdot e^{0.158 \cdot 1} = e^{0.158} \approx 1.171 \qquad \text{Using } Pe^{rt}$$

Better

Example 6 – *Solution*

cont'd

Therefore, 15.8% compounded continuously is better.

(The difference is only a tenth of a cent, but it would be more impressive for principals larger than $1 or periods longer than 1 year.)

How to Compare Interest Rates

The actual percentage increase during 1 year is called the **effective rate of interest,** the **annual percentage rate (APR),** or the **annual percentage yield (APY).**

For example, the "continuous" rate of 15.8% used above gives a return of \$1.171, and after subtracting the dollar invested, leaves a gain of 0.171 on the original \$1, which means an APR of 17.1%.

How to Compare Interest Rates

The *stated* rate (here, 15.8%) is called the **nominal** rate of interest.

That is, a nominal rate of 15.8% compounded continuously is equivalent to an APR of 17.1%.

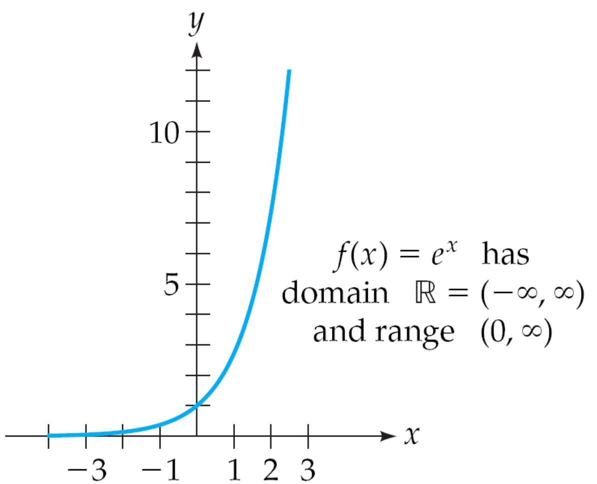
The 1993 Truth in Savings Act requires that banks always state the annual percentage rate.

The Function $y = e^x$

The Function $y = e^x$

The number *e* gives us a new exponential function $f(x) = e^x$. This function is used extensively in business, economics, and all areas of science. The table below shows the values of e^x for various values of *x*. These values lead to the graph of $f(x) = e^x$ shown at the right.

x	$y = e^x$
-3	$e^{-3} \approx 0.05$
-2	$e^{-2} \approx 0.14$
-1	$e^{-1} \approx 0.37$
0	$e^{0} = 1$
1	$e^{1} \approx 2.72$
2	$e^{2} \approx 7.39$
3	$e^{3} \approx 20.09$



Graph of $y = e^x$ is $\mathbb{R} = (-\infty, \infty)$ and range is $(0, \infty)$.

The Function $y = e^x$

Notice that e^x is never zero, and is positive for all values of x, even when x is negative.

We restate this important observation as follows:

> e to any power is positive.

The Function $y = e^x$

The graph shows the
function $f(x) = e^{kx}$ for various
values of the constant k. Fo
positive values of k
rises, an

For higher values of k the curve
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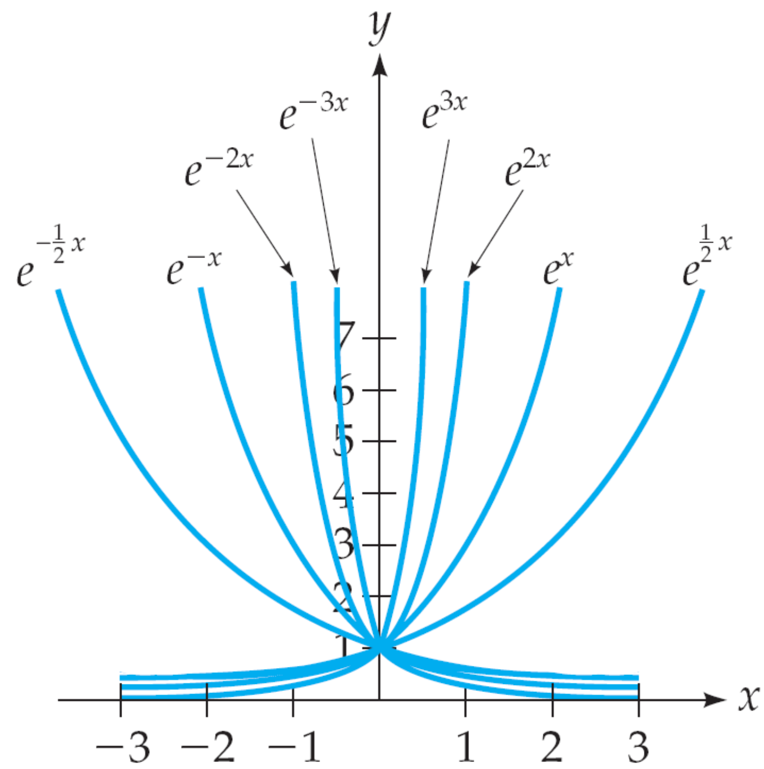

$f(x) = e^{kx}$ for various values of k.