3.1

GRAPHING USING THE FIRST DERIVATIVE

Introduction

Introduction

- In this section we will put derivatives to two major uses: **graphing** and **optimization.**

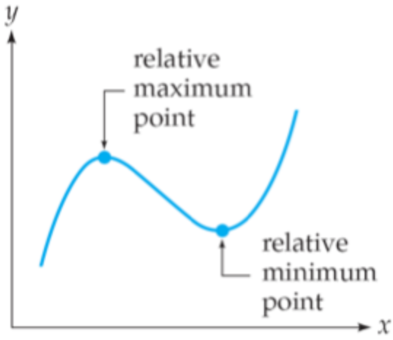
- Graphing involves using calculus to find the most important points on a curve, then sketching the curve either by hand or using a graphing calculator.

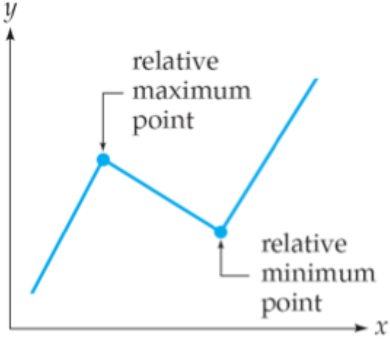
- Optimization means finding the largest or smallest values of a function (for example, maximizing profit or minimizing risk).

Relative Extreme Points and Critical Numbers

Relative Extreme Points and Critical Numbers

On a graph, a **relative maximum point** is a point that is at least as *high* as the neighboring points of the curve on either side, and a **relative minimum point** is a point that is at least as *low* as the neighboring points on either side.





Relative Extreme Points and Critical Numbers

The word “relative” means that although these points may not be the highest and lowest on the *entire* curve, they are the highest and lowest *relative to points nearby*.

A curve may have any number of relative maximum and minimum points (collectively, **relative extreme points**), even none.

Relative Extreme Points and Critical Numbers

For a function *f*, the relative extreme points may be defined more formally in terms of the values of *f*.

f has a **relative maximum value** at *c* if

fo $f(c) \geq f(x)$ *c*. bigger than

f has a **relative minimum value** at *c* if

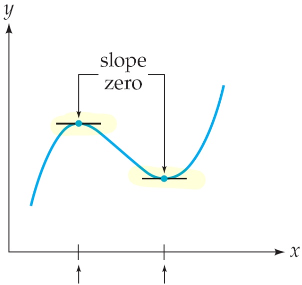
for $f(c) \leq f(x)$ *c*. lesser than

By "near *c*" we mean in some open interval containing *c.*

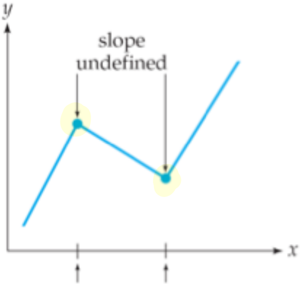
Relative Extreme Points and Critical Numbers

In the first of the two graphs below, the relative extreme points occur where the slope is *zero* (where the tangent line is horizontal), and in the second graph they occur where the slope is *undefined* (at corner points).

The *x*-coordinates of such points are called **critical numbers.**



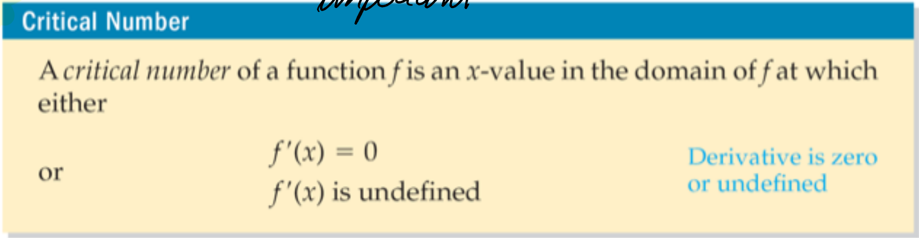
This function has two critical numbers
(where the derivative is zero).



is function also has two critical numbers
but where the derivative is undefined).

Relative Extreme Points and Critical Numbers

important



Critical Number

A *critical number* of a function f is an x-value in the domain of f at which either

$$f'(x) = 0$$

or

$$f'(x) \text{ is undefined}$$

Derivative is zero or undefined

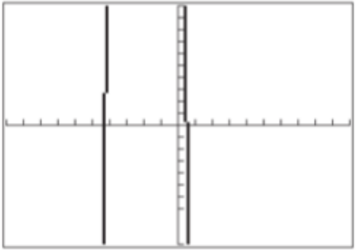
Graphing Functions

Graphing Functions

We graph a function by finding its critical numbers, making a **sign diagram** for the derivative to show the intervals of increase and decrease and the relative extreme points, and then drawing the curve on a graphing calculator or "by hand."

Obtaining a reasonable graph even with a graphing calculator requires pushing buttons, as sho unsatisfactory graph of

$f(x) = x3 - 12x2 - 60x + 36$



A "useless" graph of
$f(x) = x^3 - 12x^2 - 60x + 36$
on [−10, 10] by [−10, 10]

Example 1 – *GRAPHING A FUNCTION*

Graph the function $f(x) = x3 - 12x2 - 60x + 36$.

Solution:

Step 1: **Find critical numbers.**

$$f'(x) = 3x2 - 24x - 60$$

Derivative of $f(x) = x3 - 12x2 - 60x + 36$

$$= 3(x2 - 8x - 20)$$

$$= 3\underbrace{(x - 10)}_{\text{Zero at } x = 10}\underbrace{(x + 2)}_{\text{Zero at } x = -2} = 0$$

Factoring and setting equal to zero

Example 1 – *Solution*

cont'd

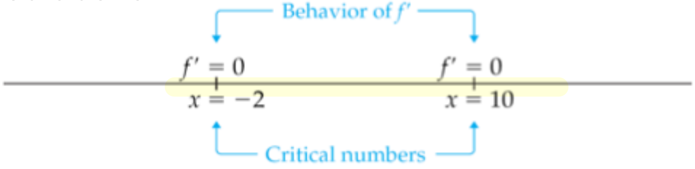
The derivative is zero at *x* = 10 and at *x* = –2, and there are no numbers at which the derivative is undefined (it is a polynomial), so the critical numbers (CNs) are

$$\text{CN}\begin{cases} x = 10 \\ x = -2 \end{cases}$$

Both are in the domain of the orginal function

Step 2: **Make a sign diagram for the derivative.**

A sign diagram for *f*′ begins with a copy of the *x*-axis with the critical numbers written below it and the behavior of *f*′ indicated above it.



Example 1 – *Solution*

cont’d

Since f' is continuous, it can change sign only at critical numbers, so f' must keep the same sign between consecutive critical numbers.

We determine the sign of f' in each interval by choosing a **test point** in each interval and substituting it into f'.

We use the factored form: $f'(x) = 3(x - 10)(x + 2)$.

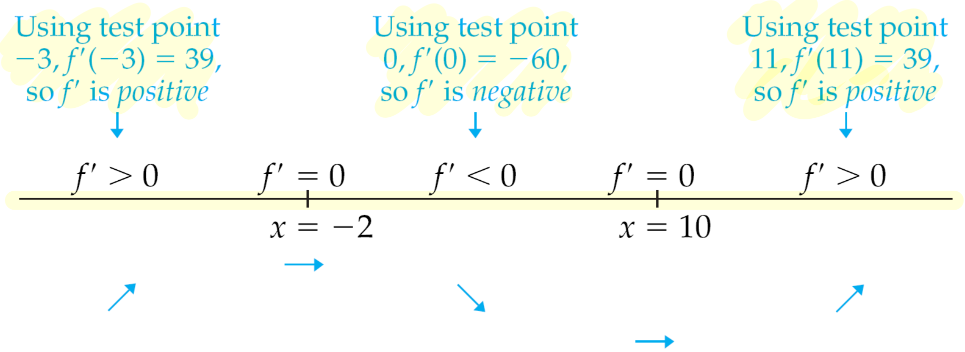
For the first interval, choosing –3 for the test point,

$$\begin{aligned} f'(-3) &= 3(-3 - 10)(-3 + 2) \\ &= 3(\text{negative})(\text{negative}) \\ &= (\text{positive}). \end{aligned}$$

Example 1 – *Solution*

cont'd

We indicate the sign of *f*′ (the slope of *f*) by arrows: ↗ for positive slope, → for zero slope, and ↘ for negative slope.



Using test point $-3, f'(-3) = 39$, so f' is *positive*

Using test point $0, f'(0) = -60$, so f' is *negative*

Using test point $11, f'(11) = 39$, so f' is *positive*

Example 1 – *Solution*

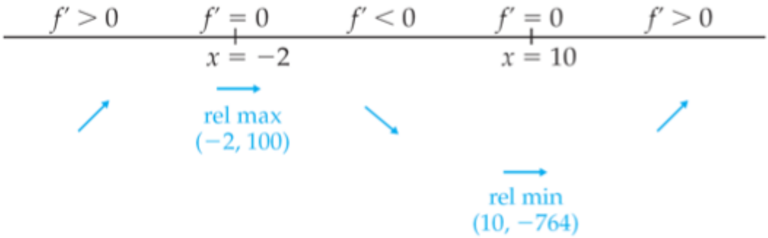
cont'd

The sign diagram shows that to the left of –2 the function increases, then between –2 and 10 it decreases, and then to the right of 10 it increases again.

Therefore, the open intervals of increase are $(-\infty, -2)$ and $(10, \infty)$, and the open interval of decrease is (–2, 10).

Example 1 – *Solution*

cont'd



Arrows ↗ → ↘ indicate a relative *maximum* point, and ↘ → ↗ arrows indicate a relative *minimum* point. We then list these points under the critical numbers.

The *y*-coordinates of the points were found by evaluating the *original* function at the *x*-coordinate: $f(-2) = 100$ and $f(10) = -764$.

Example 1 – *Solution*

cont'd

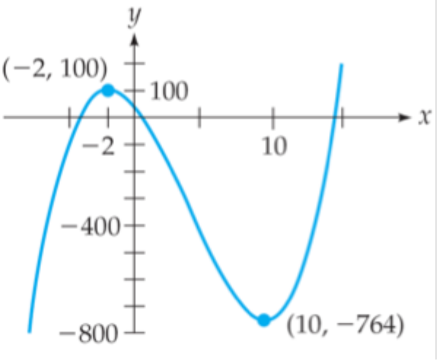
Step 3: **Sketch the graph.**

Our arrows ↗ → ↘ → ↗he general shape of the curve: going up, level, down, level, and up again. The critical numbers show that the graph should include *x*-values from before $x = –2$ to after $x = 10$, suggesting an interval such as [–10, 20]. The *y*-coordinates show that we want to go from above 100 to below –764, suggesting an interval of *y*-values such as [– 800, 200].

Example 1 – *Solution*

cont'd

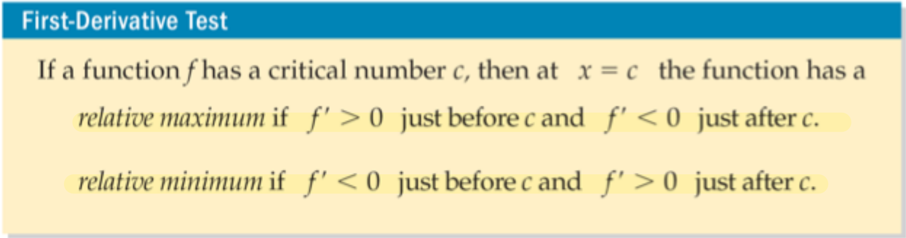
By hand, you would plot the relative maximum point (with a "cap" ∩) and the relative minimum point (with a "cup" ∪), and then draw an "up-down-up" curve through them.



First-Derivative Test for Relative Extreme Values

First-Derivative Test for Relative Extreme Values

The graphical idea from the sign diagram, that ↗ → ↘ (up, level, and down) indicates a relative *maximum* and ↘ → ↗ (down, level, and up) indicates a relative *minimum,* can be stated more formally in terms of the derivative.



First-Derivative Test

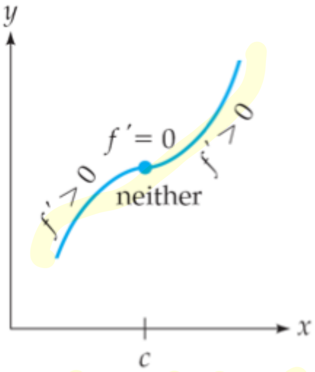
If a function *f* has a critical number *c*, then at $x = c$ the function has a

relative maximum if $f' > 0$ just before *c* and $f' < 0$ just after *c*.

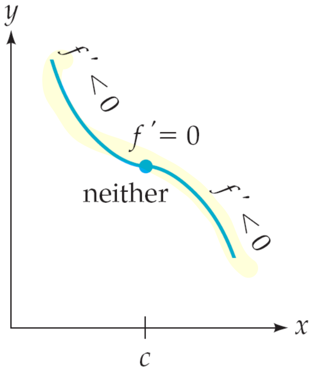
relative minimum if $f' < 0$ just before *c* and $f' > 0$ just after *c*.

First-Derivative Test for Relative Extreme Values

If the derivative has the *same* sign on both sides of *c*, then the function has *neither* a relative maximum nor a relative minimum at *x* = *c*.



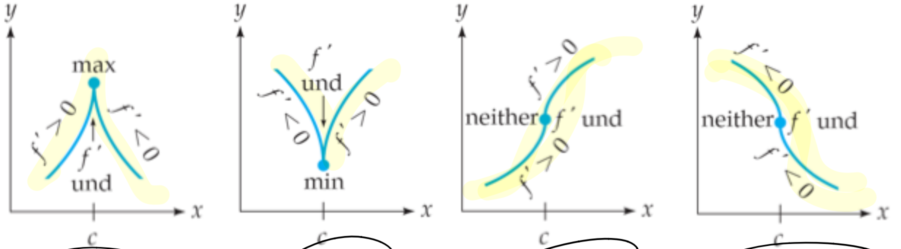
f' is positive on both sides, so f has *neither* at $x = c$.



f' is negative on both sides, so f has *neither* at $x = c$.

First-Derivative Test for Relative Extreme Values

The diagrams below show that the first-derivative test applies even at critical numbers where the derivative is *undefined* (abbreviated: *f′* und).



f' is positive then negative, so f has a relative *maximum* at $x = c$.

f' is negative then positive, so f has a relative *minimum* at $x = c$.

f' is positive on both sides, so f has *neither* at $x = c$.

f' is negative on both sides, so f has *neither* at $x = c$.

Example 2 – *GRAPHING A FUNCTION*

Graph the function $f(x) = –x4 + 4x3 – 20$.

Solution:

$f'(x) = –4x3 + 12x2$

$= –4x2(x – 3) = 0$

Differentiating

Factoring and setting equal to zero

||

, 0

Critical numbers:

$$\text{CN}\begin{cases} x = 0 \\ x = 3 \end{cases}$$

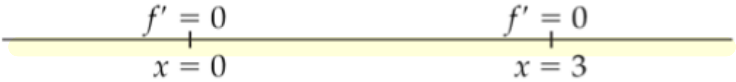
From $–4x2 = 0$

From $(x – 3) = 0$

Example 2 – *Solution*

cont'd

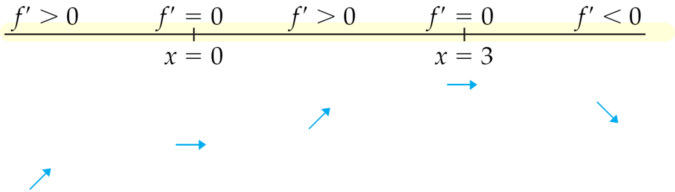
We make a sign diagram for the derivative:



Behavior of *f*

critical values

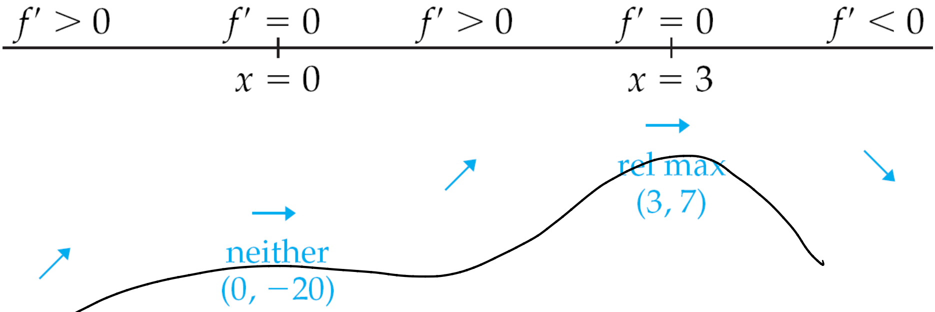
We determine the sign of $f'(x) = -4x2(x - 3)$ using test points in each interval (such as – 1 in the leftmost interval), and then add arrows



Example 2 – *Solution*

cont'd

Finally, we interpret the arrows to describe the behavior of the function, which we state under the horizontal arrows.

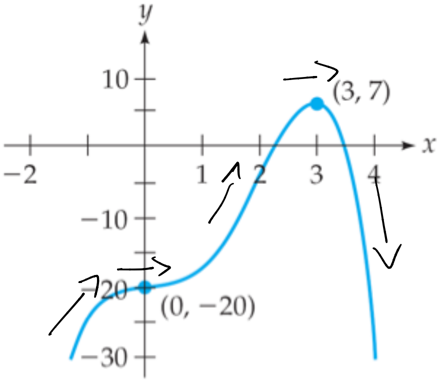


The open intervals of increase are ($-\infty$, 0) and (0, 3), and the open interval of decrease is (3, ∞).

Example 2 – *Solution*

cont'd

By hand, we would plot (0, –20) and (3, 7), and join them by a curve that goes, according to the arrows, "up-level-up-level-down."

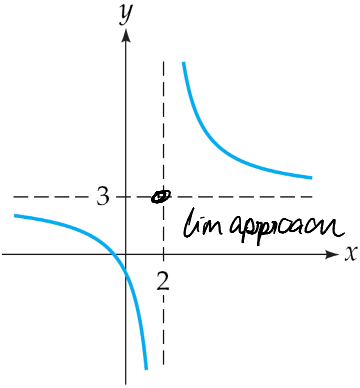


Graphing Rational Functions

Graphing Rational Functions

A rational function is a quotient of polynomials $\frac{p(x)}{q(x)}$.

We graph a rational function $f(x) = \frac{3x+2}{x-2}$ (shown on the right) with its *vertical asymptote* (where the denominator is zero and near which the curve becomes arbitrarily high or low) and *horizontal asymptote* (which the curve approaches as $x \to \infty$ or $x \to -\infty$).



Graph of $f(x) = \frac{3x+2}{x-2}$

Graphing Rational Functions

If the polynomial in the denominator is zero at *x* = *c*, by continuity it will be *near* zero for *x*-values *near c,* and reciprocals of numbers near zero are very large or very small (for example, $\frac{1}{.001}$ = 1000 and $\frac{1}{-.001}$–1000).

This leads to the following criterion for vertical asymptotes.

important

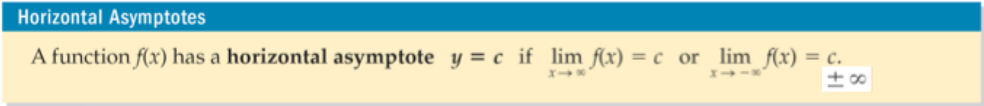
Vertical Asymptotes

A rational function $\frac{p(x)}{q(x)}$ has a **vertical asymptote** $x = c$ if $q(c) = 0$ but $p(c) \neq 0$.

Graphing Rational Functions

A function has a *horizontal* asymptote if it has a limit as

∞ ∞ $x \rightarrow$ or $x \rightarrow -$).



Horizontal Asymptotes

A function $f(x)$ has a **horizontal asymptote** $y = c$ if $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

$\pm\infty$

Graphing Rational Functions

Finding horizontal asymptotes by taking limits as *x* approaches $\pm\infty$ involves using some particular limits, each of which comes from the fact that *the reciprocal of a large number is a small number*

(for example, $\frac{1}{1000}$ = .001):

$\lim_{x\to\infty}\frac{1}{x}=0,\quad \lim_{x\to\infty}\frac{1}{x^2}=0,$ or more generally, $\lim_{x\to\infty}\frac{1}{x^n}=0$

for any positive integer *n*.

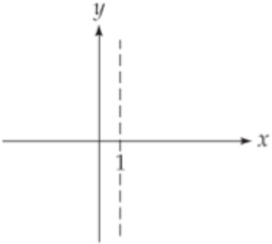
These results also hold if the 1 in the numerator is replaced by any other number, and the same results hold for limits as *x* approaches *negative* infinity.

Example 3 – *GRAPHING A FUNCTION*

Graph $f(x) = \dfrac{2x + 3}{x - 1}.$

Solution:

Since this is a rational function, we first look for vertical asymptotes. The denominator is zero at $x = 1$, and the numerator is not, so the line $x = 1$ is a vertical asymptote, which we show on the graph below.



Example 3 – *Solution*

cont'd

To find *horizontal* asymptotes, take the limit as *x* approaches ∞ or $-\infty$.

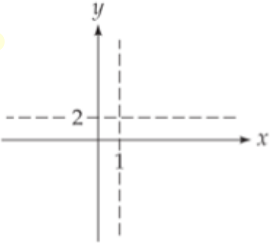
Dividing the numerator and denominator of the function by *x* and simplifying makes it easy to find the limit:

$$\lim_{x\to\infty}\frac{2x+3}{x-1}=\underbrace{\lim_{x\to\infty}\frac{\frac{2x}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{1-\frac{1}{x}}}_{\text{Dividing numerator and denominator by } x \text{ and then simplifying}}=\underbrace{\frac{2+0}{1-0}}_{\text{Since } \frac{3}{x} \text{ and } \frac{1}{x} \text{ both approach } 0}=\underbrace{\frac{2}{1}=2}_{\text{Simplifying: the limit is } 2}$$

Example 3 – *Solution*

cont'd

Since the limit is 2, $y = 2$ is a horizontal asymptote, as drawn on the right.
Letting $x \to -\infty$ gives the same limit and so the same horizontal asymptote.



Having found the asymptotes, we take the derivative to find the slope of the function.
The quotient rule gives:

$$f'(x) = \frac{(x-1)(2) - (1)(2x+3)}{(x-1)^2}$$

From $f(x) = \dfrac{2x+3}{x-1}$

$$= \frac{2x - 2 - 2x - 3}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

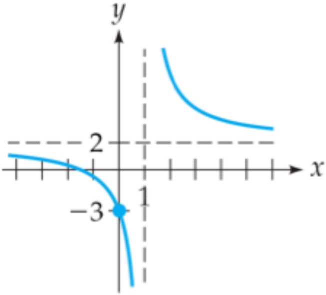
Example 3 – *Solution*

cont'd

This derivative is undefined at $x = 1$, and elsewhere it is always negative since it is a negative number over a square. Therefore, all parts of the curve slope *downward.*

The only way that the curve can slope downward everywhere and have the vertical and horizontal asymptotes we found is to have the graph shown on the right.

Evaluating at $x = 0$ and plotting the y–intercept $(0, –3)$ also helps.



Graph of $f(x) = \dfrac{2x + 3}{x - 1}$

3

Further Applications of Derivatives

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3

Further Applications of Derivatives

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3.2 GRAPHING USING THE FIRST AND SECOND DERIVATIVES

Introduction

Introduction

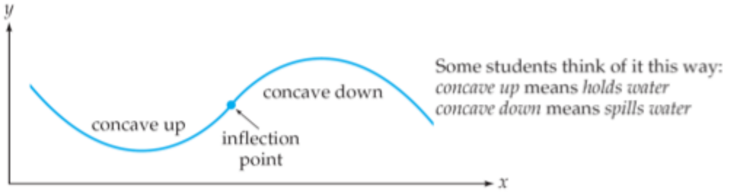
- In this section we will use the *second* derivative to find the **concavity** or **curl** of the curve, and to define the important concept of **inflection point.**

- The second derivative also gives us a very useful way to distinguish between maximum and minimum points of a curve.

Concavity and Inflection Points

Concavity and Inflection Points

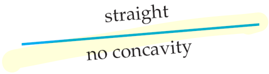
- A curve that curls upward is said to be **concave up,** and a curve that curls downward is said to be **concave down.**
- A point where the concavity *changes* (from up to down or down to up) is called an **inflection point.**



Concavity shows how a curve *curls* or *bends* away from straightness.

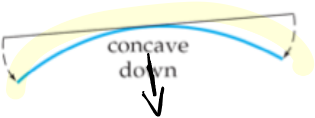
Concavity and Inflection Points

A straight line (with any slope) has *no concavity.*



However, bending the two ends *upward* makes it *concave up,*

and bending the two ends *downward* makes it *concave down.*



As these pictures show, a curve that is concave *up* lies *above* its tangent, while a curve that is concave *down* lies *below* its tangent (except at the point of tangency).

Concavity and Inflection Points

Concavity and Inflection Points

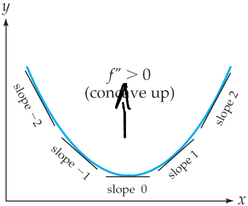
important

On an interval:

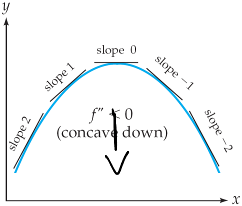
$f'' > 0$ means that f is *concave up* (curls upward).

$f'' < 0$ means that f is *concave down* (curls downward).

An *inflection point* is where the concavity *changes* (f'' must be zero or undefined).



$f'' > 0$ means that the slope is increasing, so f is *concave up*.



$f'' < 0$ means that the slope is decreasing, so f is *concave down*.

EXAMPLE 1 – *GRAPHING AND INTERPREATING A COMPANY'S ANNUAL PROFIT FUNCTION*

A company's annual profit after x years is $f(x) = x3 - 9x2 + 24x$ million dollars (for $x \geq 0$). Graph this function, showing all relative extreme points and inflection points. Interpret the inflection points.

Solution :

$$f'(x) = 3x2 - 18x + 24$$

1. factor

Differentiating

$$= 3(x2 - 6x + 8)$$

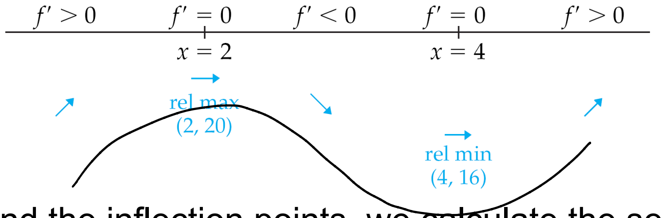
$$= 3(x - 2)(x - 4)$$

Factoring

Example 1 – *Solution*

cont'd

The critical numbers are $x = 2$ and $x = 4$, and the sign diagram for *f*′ (found in the usual way) is



$f' > 0$ $f' = 0$ $f' < 0$ $f' = 0$ $f' > 0$

$x = 2$ $x = 4$

rel max
(2, 20)

rel min
(4, 16)

To find the inflection points, we calculate the second derivative:

$$f''(x) = 6x - 18 = 6(x - 3)$$ Differentiating $f'(x) = 3x2 - 18x + 24$

Example 1 – *Solution*

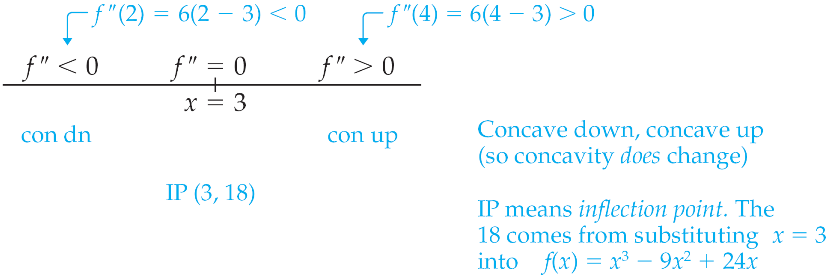
cont’d

This is zero at $x = 3$, which we enter on a sign diagram for the *second* derivative.

$f'' = 0$ ← Behavior of f''

$x = 3$ ← Where f'' is zero or undefined

We use test points to determine the sign of $f''(x) = 6(x - 3)$ on either side of 3, just as we did for the first derivative.



$f''(2) = 6(2 - 3) < 0$

$f''(4) = 6(4 - 3) > 0$

$f'' < 0$ $\quad f'' = 0$ $\quad f'' > 0$

$x = 3$

con dn $\quad$ con up

Concave down, concave up (so concavity *does* change)

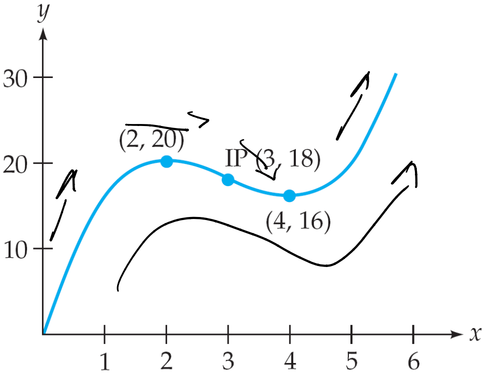
IP (3, 18)

IP means *inflection point*. The 18 comes from substituting $x = 3$ into $f(x) = x^3 - 9x^2 + 24x$

Example 1 – *Solution*

cont'd

By hand, we would plot the relative maximum (∩), minimum (∪), and inflection point and sketch the curve according to the sign diagrams, being sure to show the concavity changing at the inflection point.



Example 1 – *Solution*

cont'd

Interpretation of the inflection point:

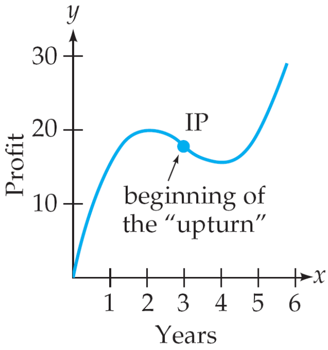
Observe what the graph shows—that the company's profit increased (up to year 2), then decreased (up to year 4),and then increased again.

The inflection point at $x = 3$ is where the profit *first began to show signs of improvement.*

Example 1 – *Solution*

cont'd

It marks the end of the period of increasingly steep decline and the first sign of an "upturn," where a clever investor might begin to "buy in." See the graph below.



Inflection Points in the Real World

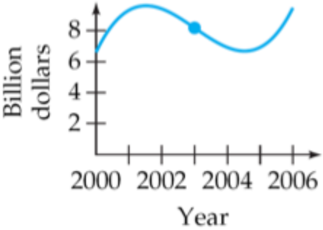
Inflection Points in the Real World

Inflection points occur in many everyday situations.

The function in Example 1, while constructed for ease of calculation, is essentially the graph of net income for AT&T over recent years, shown below.

The inflection point represents the first sign of the upturn in AT&T's net income, which occurred in 2003.

Source: Standard & Poor's



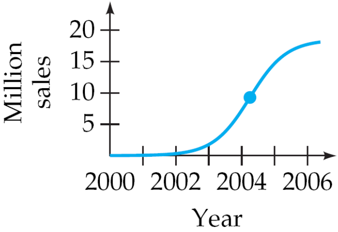
Inflection Points in the Real World

The graph below shows the annual sales of portable MP3 players.

The inflection point, occurring between 2004 and 2005, marks the end of the period of increasingly rapid sales growth and first sign of approaching market saturation.

This is when savvy manufacturers might begin to curtail new investment in MP3 production.

Source: Consumer USA 2008

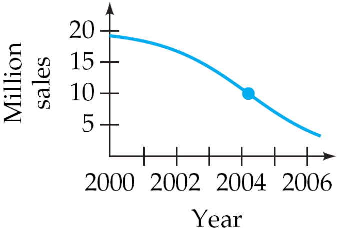


Inflection Points in the Real World

The graph below shows the annual sales of analog cameras a few years after the introduction of digital cameras.

The inflection point, occurring in 2004, marks the end of the increasingly steep sales decline and the first sign of steadying but lower sales.

Source: Euromonitor



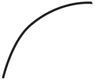
Inflection Points in the Real World

Distinguish carefully between slope and concavity: *slope* measures *steepness*, whereas *concavity* measures *curl.* All combinations of slope and concavity are possible.

A graph may be

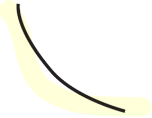
Increasing and concave *up*
($f' > 0, f'' > 0$), such as

Increasing and concave *down*
($f' > 0, f'' < 0$), such as

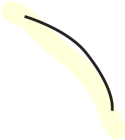


Inflection Points in the Real World

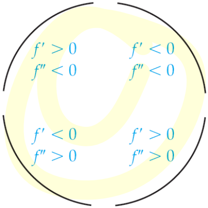
Decreasing and concave *up*
($f' < 0$, $f'' > 0$), such as



Decreasing and concave *down*
($f' < 0$, $f'' < 0$), such as



The four quarters of a circle illustrate all four possibilities, as shown at the right.



Example 2 – *GRAPHING A FRACTIONAL POWER FUNCTION*

Graph $f(x) = 18x1/3$.

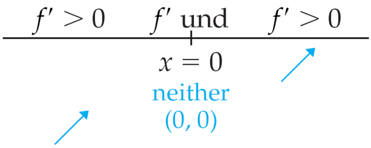
Solution :

The derivative is

$$f'(x) = 6x^{-2/3} = \frac{6}{\sqrt[3]{x^2}}$$

Undefined at $x = 0$
(zero denominator)

The sign diagram for f' is



f' is undefined at $x = 0$ and positive on either side (using test points)

Example 2 – *Solution*

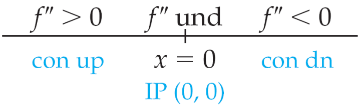
cont’d

The *second* derivative is

$$f''(x) = -4x^{-5/3} = \frac{-4}{\sqrt[3]{x^5}}$$

Also undefined at $x = 0$

The sign diagram for *f''* is



$f'' > 0$ f'' und $f'' < 0$

con up $x = 0$ con dn

IP (0, 0)

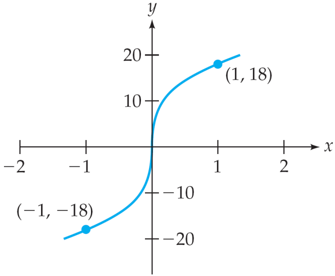
Concavity is different on either side of $x = 0$ (using test points), so there *is* an inflection point at $x = 0$

Example 2 – *Solution*

cont'd

Based on this information, we may graph the function by hand:

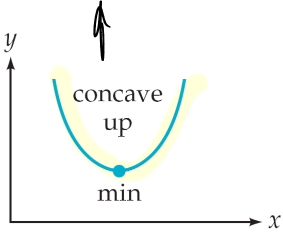
By hand, we use the sign diagrams to draw the curve to the *left* of $x = 0$ as *increasing* and concave *up,* and to the right of $x = 0$ as increasing and concave *down,* with the two parts meeting at the origin. The scale comes from calculating the points(1, 18) and (–1, –18).



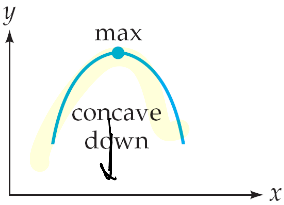
Second-Derivative Test

Second-Derivative Test

Determining whether a twice-differentiable function has a relative maximum or minimum at a critical number is merely a question of concavity: concave *up* means a relative *minimum,* and concave *down* means a relative *maximum.*



Concave *up* at a critical number: relative *minimum*.



Concave *down* at a critical number: relative *maximum*.

Since the second derivative determines concavity, we have the *second-derivative test*.

Second-Derivative Test

important

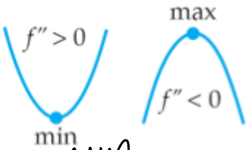
Second-Derivative Test for Relative Extreme Points

If $x = c$ is a critical number of f at which f'' is defined, then

$f''(c) > 0$ means that f has a relative *minimum* at $x = c$.

$f''(c) < 0$ means that f has a relative *maximum* at $x = c$.

To use the second-derivative test, first find all critical numbers, substitute each into the second derivative (if possible), and determine the sign of the result: a *positive* result means a *minimum* at the critical number, and a *negative* result means a *maximum.*



concave down

concave up

Example 4 – *USING THE SECOND-DERIVATIVE TEST*

Use the second-derivative test to find all relative extreme points of

$f(x) = x3 - 9x2 + 24x$.

Solution :

$$
\begin{aligned}
f'(x) &= 3x^2 - 18x + 24 && \text{The derivative} \\
&= 3(x^2 - 6x + 8) = 3(x - 2)(x - 4) && \text{Factoring} \\
\text{CN} &\begin{cases} x = 2 \\ x = 4 \end{cases} && \text{Critical numbers}
\end{aligned}
$$

Example 4 – *Solution*

cont'd

We substitute each critical number into $f''(x) = 6x - 18$.

At $x = 2$: $f''(2) = 6 \cdot 2 - 18 = -6$ (negative)

$f''(x) = 6x - 18$ at $x = 2$

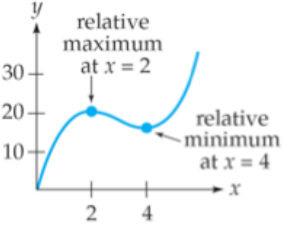
Therefore, f has a relative *maximum* at $x = 2$.

At $x = 4$: $f''(4) = 6 \cdot 4 - 18 = 6$ (positive)

$f''(x) = 6x - 18$ at $x = 4$

Therefore, f has a relative *minimum* at $x = 4$.

The relative maximum at $x = 2$ and the relative minimum at $x = 4$ are exactl what we found before when we graphed this function as sho on the left.

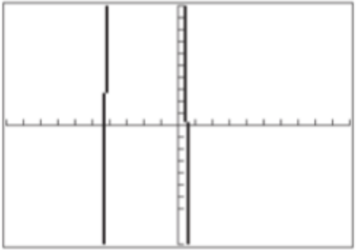


Graphing Functions

We graph a function by finding its critical numbers, making a **sign diagram** for the derivative to show the intervals of increase and decrease and the relative extreme points, and then drawing the curve on a graphing calculator or "by hand."

Obtaining a reasonable graph even with a graphing calculator requires pushing buttons, as sho unsatisfactory graph of

f(*x*) = *x*3 – 12*x*2 – 60*x* + 36



A "useless" graph of
$f(x) = x^3 - 12x^2 - 60x + 36$
on $[-10, 10]$ by $[-10, 10]$

3.5 OPTIMIZING LOT SIZE AND HARVEST SIZE

Introduction

Introduction

- In this section we discuss two important applications of optimization, one economic and one ecological.
- The first concerns the most efficient way for a business to order merchandise (or for a manufacturer to produce merchandise), and the second concerns the preservation of animal populations that are harvested by people.
- Either of these applications may be read independently of the other.

Minimizing Inventory Costs

Minimizing Inventory Costs

A business encounters two kinds of costs in maintaining inventory: **storage costs** (warehouse and insurance costs for merchandise not yet sold) and **reorder costs** (delivery and bookkeeping costs for each order).

For example, if a furniture store expects to sell 250 sofas in a year, it could order all 250 at once (incurring high storage costs), or it could order them in many small lots, say 50 orders of five each, spaced throughout the year (incurring high reorder costs).

Obviously, the best order size (or **lot size**) is the one that minimizes the total of storage plus reorder costs.

Example 1 – *MINIMIZING INVENTORY COSTS*

A furniture showroom expects to sell 250 sofas a year. Each sofa costs the store $300, and there is a fixed charge of $500 per order. If it costs $100 to store a sofa for a year, how large should each order be and how often should orders be placed to minimize inventory costs?

Solution:

Let

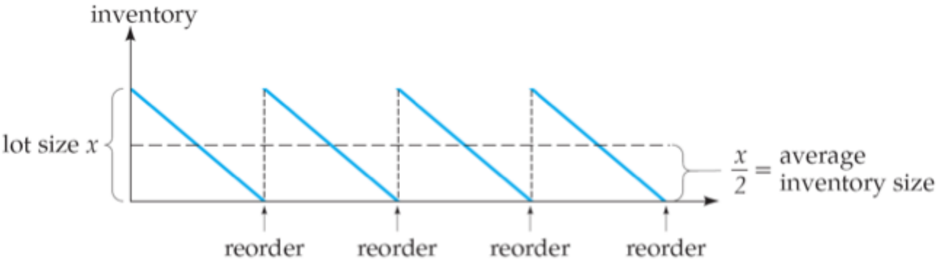
x = lot size Number of sofas in each order

Example 1 – *Solution*

cont'd

Storage Costs:

If the sofas sell steadily throughout the year, and if the store reorders *x* more whenever the stock runs out, then its inventory during the year looks like the following graph.



Example 1 – *Solution*

cont'd

Notice that the inventory level varies from the lot size x down to zero, with an average inventory of $x/2$ sofas throughout the year.

Because it costs \$100 to store a sofa for a year, the total (annual) storage costs are

$$\begin{pmatrix}\text{Storage}\\ \text{costs}\end{pmatrix} = \begin{pmatrix}\text{Storage}\\ \text{per item}\end{pmatrix}\begin{pmatrix}\text{Average num-}\\ \text{ber of items}\end{pmatrix}$$

$$= 100 \cdot \frac{x}{2}$$

$$= 50x$$

Example 1 – *Solution*

cont'd

Reorder Costs:

Each sofa costs \$300, so an order of lot size *x* costs 300*x*, plus the fixed order charge of \$500:

$$\begin{pmatrix}\text{Cost}\\ \text{per order}\end{pmatrix} = 300x + 500$$

The yearly supply of 250 sofas, with *x* sofas in each order, requires $\frac{250}{x}$ orders. (For example, 250 sofas at 5 per order require $\frac{250}{5} = 50$ers.)

Therefore, the yearly reorder costs are

$$\begin{pmatrix}\text{Reorder}\\ \text{costs}\end{pmatrix} = \begin{pmatrix}\text{Cost}\\ \text{per order}\end{pmatrix} \cdot \begin{pmatrix}\text{Number}\\ \text{of orders}\end{pmatrix} = (300x + 500) \cdot \left(\frac{250}{x}\right)$$

Example 1 – *Solution*

cont’d

Total Cost:

$C(x)$ is storage costs plus reorder costs:

$$C(x) = \begin{pmatrix}\text{Storage}\\ \text{costs}\end{pmatrix} + \begin{pmatrix}\text{Reorder}\\ \text{costs}\end{pmatrix}$$

$$= 100\frac{x}{2} + (300x + 500)\left(\frac{250}{x}\right)$$

Using the storage and reorder costs found earlier

$$= 50x + 75{,}000 + 125{,}000x^{-1}$$

Simplifying

To minimize $C(x)$, we differentiate:

$$C'(x) = 50 - 125{,}000x^{-2}$$

$$= 50 - \frac{125{,}000}{x^2}$$

Differentiating $C = 50x + 75{,}000 + 125{,}000x$–1

Example 1 – *Solution*

cont'd

$$50 - \frac{125{,}000}{x^2} = 0$$

Setting the derivative equal to zero

$$50x^2 = 125{,}000$$

Multiplying by *x*2 and adding 125,000 to each side

$$x^2 = \frac{125{,}000}{50} = 2500$$

Dividing each *x*2 side by 50

$$x = 50$$

Taking square roots (*x* > 0) gives lot size 50

$$C''(x) = 250{,}000x^{-3} = 250{,}000\frac{1}{x^3}$$

C'' is positive, so *C* is minimized at *x* = 50

At 50 sofas per order, the yearly 250 will require $\frac{250}{50}$ = 5 orders.

Therefore: Lot size is 50 sofas, with orders placed five times a year.

Modifications and Assumptions

Modifications and Assumptions

If the number of orders per year is not a whole number, say 7.5 orders per year, we just interpret it as 15 orders in 2 years, and handle it accordingly.

We made two major assumptions in Example 1. We assumed that there was a steady demand, and that orders were equally spaced throughout the year.

These are reasonable assumptions for many products, while for seasonal products such as bathing suits or winter coats, separate calculations can be done for the “on” and “off” seasons.

Production Runs

Production Runs

Similar analysis applies to manufacturing.

For example, if a book publisher can estimate the yearly demand for a book, she may print the yearly total all at once, incurring high storage costs, or she may print them in several smaller runs throughout the year, incurring setup costs for each run.

Here the setup costs for each printing run play the role of the reorder costs for a store.

Example 2 – *MINIMIZING INVENTORY COSTS FOR A PUBLISHER*

A publisher estimates the annual demand for a book to be 4000 copies. Each book costs $8 to print, and setup costs are $1000 for each printing. If storage costs are $2 per book per year, find how many books should be printed per run and how many printings will be needed if costs are to be minimized.

Solution:

Let

x = the number of books in each run

Example 2 – *Solution*

cont'd

Storage Costs

As in Example 1, an average of $\frac{x}{2}$ books are stored throughout the year, at a cost of $2 each, so annual storage costs are

$$\begin{pmatrix}\text{Storage}\\\text{costs}\end{pmatrix} = \left(\frac{x}{2}\right)\cdot 2 = x$$

Production Costs

The cost per run is

$$\begin{pmatrix}\text{Costs}\\\text{per run}\end{pmatrix} = 8x + 1000$$

x books at $8 each, plus $1000 setup costs

Example 2 – *Solution*

cont'd

The 4000 books at *x* books per run will require $\frac{4000}{x}$ runs. Therefore, production costs are

$$\begin{pmatrix}\text{Production}\\ \text{costs}\end{pmatrix} = (8x + 1000)\left(\frac{4000}{x}\right)$$

Cost per run times number of runs

Total Cost

The total cost is storage costs plus production costs:

$$\begin{aligned} C(x) &= x + (8x + 1000)\left(\frac{4000}{x}\right) \\ &= x + 32{,}000 + 4{,}000{,}000x^{-1} \end{aligned}$$

Example 2 – *Solution*

cont’d

We differentiate, set the derivative equal to zero, and solve, just as before (omitting the details), obtaining $x = 2000$.

The second-derivative test will show that costs are minimized at 2000 books per run.

The 4000 books require $\frac{4000}{2000}$ = 2 printings.

Therefore, the publisher should:

Print 2000 books per run, with two printings.

Maximum Sustainable Yield

Maximum Sustainable Yield

The next application involves industries such as fishing, in which a naturally occurring animal population is “harvested.”

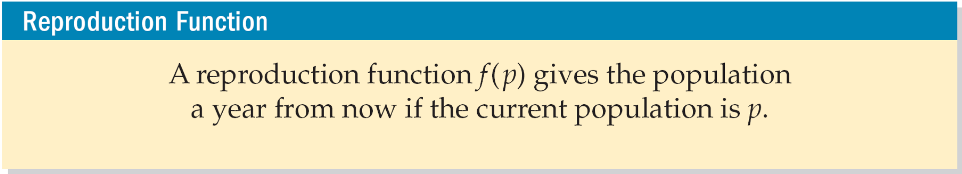
Harvesting some of the population means more food and other resources for the remaining population so that it will expand and replace the harvested portion.

But taking too large a harvest will kill off the animal population.

We want to find the **maximum sustainable yield,** the largest amount that may be harvested year after year and still have the population return to its previous level the following year.

Maximum Sustainable Yield

For some animals one can determine a **reproduction function** *f*(*p*) which gives the expected population a year from now if the present population is *p.*



Reproduction Function

A reproduction function $f(p)$ gives the population a year from now if the current population is p.

Maximum Sustainable Yield

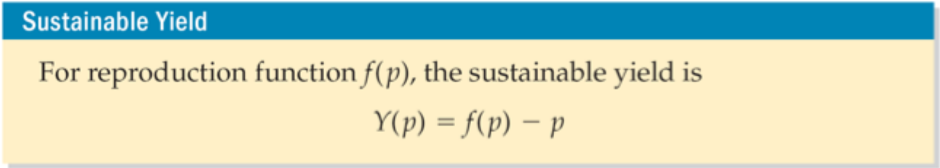
Suppose that we have a reproduction function *f* and a current population of size *p*, which will therefore grow to size *f*(*p*) next year. The *amount of growth* in the population during that year is

$$\begin{pmatrix}\text{Amount}\\ \text{of growth}\end{pmatrix} = \underset{\substack{\uparrow\\ \text{Next year's population}}}{f(p)} - \underset{\substack{\uparrow\\ \text{Current population}}}{p}$$

Harvesting this amount removes only the *growth,* returning the population to its former size *p.*

Maximum Sustainable Yield

The population will then repeat this growth, and taking the same harvest $f(p) - p$ will cause this situation to repeat itself year after year. The quantity $f(p) - p$ is called the sustainable yield.



Sustainable Yield

For reproduction function $f(p)$, the sustainable yield is

$$Y(p) = f(p) - p$$

Maximum Sustainable Yield

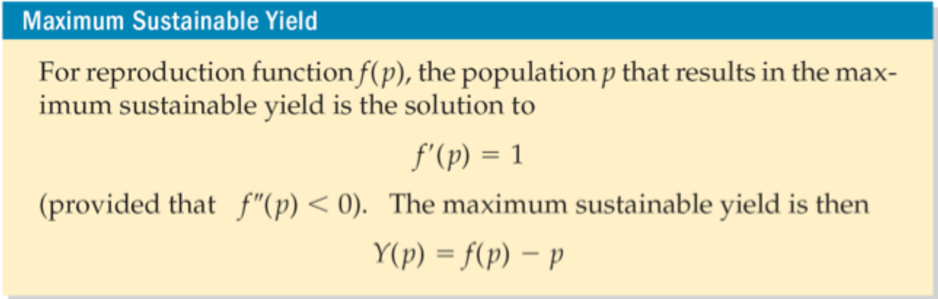
We want the population size p that maximizes the sustainable yield $Y(p)$. To maximize $Y(p)$, we set its derivative equal to zero:

$Y'(p) = f'(p) - 1 = 0$ Derivative of $Y = f(p) - p$

$f'(p) = 1$ Solving for $f'(p)$

For a given reproduction function $f(p)$, we find the maximum sustainable yield by solving this equation (provided that the second-derivative test gives $Y''(p) = f''(p) < 0$).

Maximum Sustainable Yield



Maximum Sustainable Yield

For reproduction function $f(p)$, the population p that results in the maximum sustainable yield is the solution to

$$f'(p) = 1$$

(provided that $f''(p) < 0$). The maximum sustainable yield is then

$$Y(p) = f(p) - p$$

Once we calculate the population *p* that gives the maximum sustainable yield, we wait until the population reaches this size and then harvest, year after year, an amount *Y*(*p*).

Example 3 – *FINDING MAXIMUM SUSTAINABLE YIELD*

The reproduction function for the American lobster in an East Coast fishing area is *f*(*p*) = –0.02*p*2 + 2*p* (where *p* and *f*(*p*) are in thousands). Find the population *p* that gives the maximum sustainable yield and find the size of the yield.

Solution:

We set the derivative of the reproduction function equal to 1:

$$f'(p) = -0.04p + 2 = 1$$

$$-0.04p = -1$$

ting 2 from each side

$$p = \frac{-1}{-0.04} = 25$$

Example 3 – *Solution*

cont'd

The second derivative is $f''(p) = -0.04$, which is negative, showing that $p = 25$ (thousand) is the population that gives the maximum sustainable yield.

The actual yield is found from the yield function

$f(p) = Y(p) - p$:

$$Y(p) = -0.02p2 + 2p - p$$

$$= 0.02p2 + p$$

$$\underbrace{\hphantom{0.02p2 + p}}_{f(p)}$$

$$Y(25) = -0.02(25)2 + 25$$

Evaluating at $p = 25$ (thousand)

The population size for the maximum sustainable yield is 25,000, and the yield is 12,500 lobsters.

25,000 from $p = 25$

3

Further Applications of Derivatives

Stockbyte/Jupiterimages

3.7 Differentials, Approximations, and Marginal Analysis

Introduction

Introduction

- We often want to know how one change affects another. For example, a manager might want to estimate the added profit from a small increase in production.

- In this section, we estimate such changes by using the tangent line to approximate the function near the point of tangency.
- In Chapter 2 we defined the derivative by approximating the tangent-line slope by secant-line slopes, and now we reverse our viewpoint and estimate secant-line changes by tangent-line changes.

Differentials

Differentials

The equation $x2 + y2 = 25$ defines a circle.

While a circle is not the graph of a function the top half by itself defines a function, as does the bottom half by itself.

To find these two functions, we solve $x2 + y2 = 25$ for y:

$$y2 = 25 - x2$$

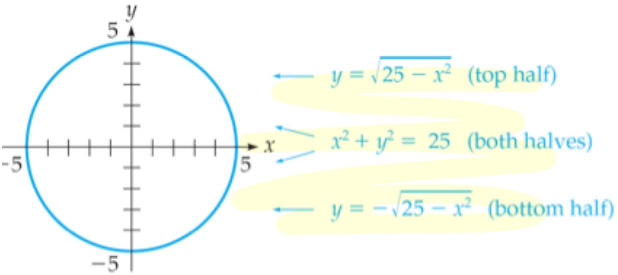
Subtracting $x2$ from each side of $x2 + y2 = 25$

$$y = \pm\sqrt{25 - x^2}$$

Plus or minus since when squared either one gives $25 - x2$

Implicit Differentiation

The *positive* square root defines the top half of the circle (where *y* is positive), and the *negative* square root defines the bottom half (where *y* is negative). The equation
$x2 + y2 = 25$ defines *both* functions at the same time.



Implicit Differentiation

To find the slope anywhere on the circle, we could differentiate the “top” and “bottom” functions separately.

However, it is easier to find both answers at once by differentiating *implicitly,* that is by differentiating both sides of the equation *x*2 + *y*2 = 25 with respect to *x*.

Remember, however, that *y* is a *function* of *x*, so differentiating *y*2 means differentiating a *function* squared, which requires the Generalized Power Rule:

$$\frac{d}{dx} y^n = n \cdot y^{n-1} \frac{dy}{dx}$$

Example 1 – *DIFFERENTIATING IMPLICITLY*

Use implicit differentiation to find $\frac{dy}{dx}$ when $x2 + y2 = 25$.

Solution:

We differentiate both sides of the equation with respect to x:

$$\underbrace{\frac{d}{dx}x^2}_{\downarrow} + \underbrace{\frac{d}{dx}y^2}_{\downarrow} = \underbrace{\frac{d}{dx}25}_{\downarrow}$$

Differentiating $x^2 + y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0$$

Using the Generalized Power Rule on y^2

Example 1 – *Solution*

cont'd

Solving for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x$$

Subtracting 2*x*

$$\frac{dy}{dx} = -\frac{x}{y}$$

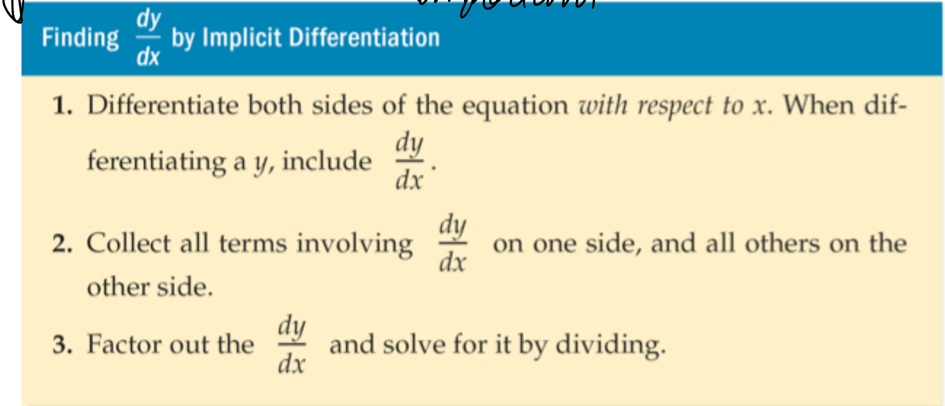
Canceling the 2's
and dividing by y

Therefore, $\frac{dy}{dx} = -\frac{x}{y}$ when *x* and *y* are related by
$x2 + y2 = 25$.

Implicit Differentiation

Implicit differentiation involves three steps.

important!



Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*. When differentiating a y, include $\frac{dy}{dx}$.
2. Collect all terms involving $\frac{dy}{dx}$ on one side, and all others on the other side.
3. Factor out the $\frac{dy}{dx}$ and solve for it by dividing.

Example 4 – *FINDING AND EVALUATING AN IMPLICIT DERIVATIVE*

For *y*4 + *x*4 – 2*x*2*y*2 = 9

a. find $\frac{dy}{dx}$

b. evaluate it at *x* = 2, *y* = 1

Solution:

$$4y^3\frac{dy}{dx} + 4x^3 - 4xy^2 - 4x^2y\frac{dy}{dx} = 0$$

Differentiating with respect to *x*, putting constants first

$$4y^3\frac{dy}{dx} - 4x^2y\frac{dy}{dx} = -4x^3 + 4xy^2$$

Collecting *dy*/*dx* terms on the left, others on the right

$$(4y^3 - 4x^2y)\frac{dy}{dx} = -4x^3 + 4xy^2$$

Factoring out $\frac{dy}{dx}$

Example 1 – *Solution*

cont'd

$$\frac{dy}{dx} = \frac{-4x^3 + 4xy^2}{4y^3 - 4x^2y}$$

← Answer for part (a)

Dividing by $4y^3 - 4x^2y$ to solve for dy/dx

$$= \frac{-x^3 + xy^2}{y^3 - x^2y}$$

Dividing by 4

$$\frac{dy}{dx} = \frac{-(2)^3 + (2)(1)^2}{(1)^3 - (2)^2(1)} = \frac{-6}{-3} = 2$$

Evaluating at $x = 2, y = 1$ gives the answer for part (b)

Implicit Differentiation

Note that in the example above the given point *is* on the curve, since $x = 2$ and $y = 1$ satisfy the original equation:

$$14 + 24 – 2 \cdot 22 \cdot 1 = 1 + 16 – 8 = 9$$

In economics, a **demand equation** is the relationship between the price p of an item and the quantity x that consumers will demand at that price.

Related Rates

Related Rates

Sometimes *both* variables in an equation will be functions of a *third* variable, usually *t* for time.

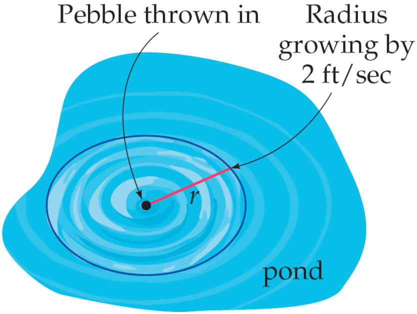
For example, for a seasonal product such as winter coats, the price *p* and weekly sales *x* will be related by a demand equation, and both price *p* and quantity *x* will depend on the time of year.

Differentiating both sides of the demand equation with respect to time *t* will give an equation relating the derivatives *dp*/*dt* and *dx*/*dt*.

Such “related rates” equations show how fast one quantity is changing relative to another. First, an “everyday” example.

Example 6 – *FINDING RELATED RATES*

A pebble thrown into a pond causes circular ripples to radiate outward. If the radius of the outer ripple is growing by 2 feet per second, how fast is the area of its circle growing at the moment when the radius is 10 feet?



Example 6 – *Solution*

The formula for the area of a circle is $A = \pi r2$.

Both the area A and the radius r of the circle increase with time, so both are functions of t.

We are told that the radius is increasing by 2 feet per second ($dr/dt = 2$), and we want to know how fast the area is changing (dA/dt).

Example 6 – *Solution*

cont'd

To find the relationship between *dA*/*dt* and *dr*/*dt*, we differentiate both sides of *A* = π*r*2 with respect to *t*.

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

From $A = \pi r^2$, writing the 2 before the π

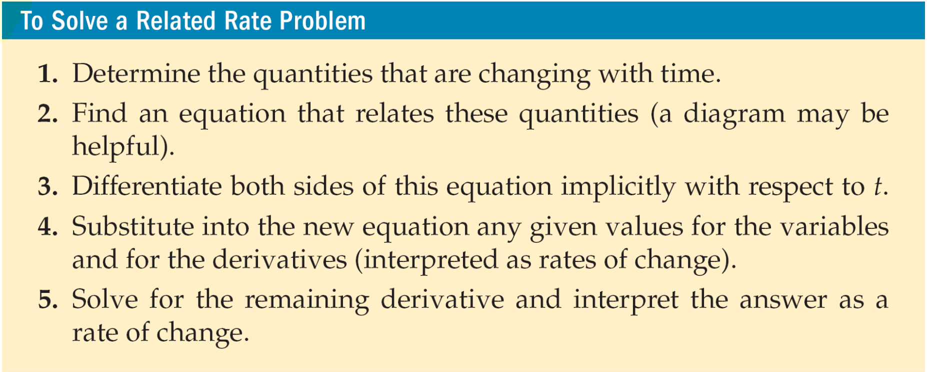
$$\frac{dA}{dt} = 2\pi \cdot \underbrace{10}_{r} \cdot \underbrace{2}_{\frac{dr}{dt}}$$

Substituting $r = 10$ and $\frac{dr}{dt} = 2$

$$= 40\pi \approx \underbrace{125.6}_{\text{Using } \pi \approx 3.14}$$

Therefore, at the moment when the radius is 10 feet, the area of the circle is growing at the rate of about 126 square feet per second.

Related Rates



To Solve a Related Rate Problem

1. Determine the quantities that are changing with time.
2. Find an equation that relates these quantities (a diagram may be helpful).
3. Differentiate both sides of this equation implicitly with respect to t.
4. Substitute into the new equation any given values for the variables and for the derivatives (interpreted as rates of change).
5. Solve for the remaining derivative and interpret the answer as a rate of change.