3 Further Applications of Derivatives

3.6 IMPLICIT DIFFERENTIATION AND RELATED RATES

Introduction

Introduction

- A function written in the form $y = f(x)$ is said to be defined *explicitly*, meaning that y is defined by a rule or *formula* $f(x)$ *in x alone*.

- A function may instead be defined **implicitly,** meaning that y is defined by an *equation in x and y.*

- In this section we will see how to differentiate such **implicit functions** when ordinary "explicit" differentiation is difficult or impossible.

- We will then use implicit differentiation to find rates of change.

Implicit Differentiation

Implicit Differentiation

The equation $x^2 + y^2 = 25$ defines a circle.

While a circle is not the graph of a function the top half by itself defines a function, as does the bottom half by itself.

To find these two functions, we solve $x^2 + y^2 = 25$ for y:

$$y^2 = 25 - x^2$$

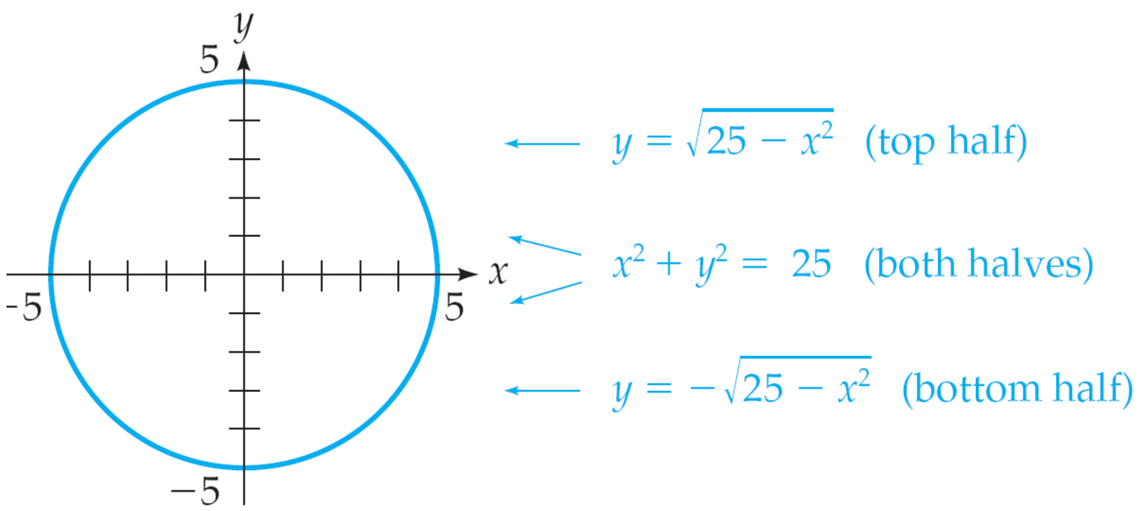
Subtracting x^2 from each side of $x^2 + y^2 = 25$

$$y = \pm\sqrt{25 - x^2}$$

Plus or minus since when squared either one gives $25 - x^2$

Implicit Differentiation

The *positive* square root defines the top half of the circle (where *y* is positive), and the *negative* square root defines the bottom half (where *y* is negative). The equation $x^2 + y^2 = 25$ defines *both* functions at the same time.



Implicit Differentiation

To find the slope anywhere on the circle, we could differentiate the “top” and “bottom” functions separately.

However, it is easier to find both answers at once by differentiating *implicitly,* that is by differentiating both sides of the equation $x^2 + y^2 = 25$ with respect to x.

Remember, however, that y is a *function* of x, so differentiating y^2 means differentiating a *function* squared, which requires the Generalized Power Rule:

$$\frac{d}{dx} y^n = n \cdot y^{n-1} \frac{dy}{dx}$$

Example 1 – *DIFFERENTIATING IMPLICITLY*

Use implicit differentiation to find $\frac{dy}{dx}$ when $x^2 + y^2 = 25$.

Solution:

We differentiate both sides of the equation with respect to x:

$$\underbrace{\frac{d}{dx}x^2}_{\downarrow} + \underbrace{\frac{d}{dx}y^2}_{\downarrow} = \underbrace{\frac{d}{dx}25}_{\downarrow}$$

Differentiating $x^2 + y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0$$

Using the Generalized Power Rule on y^2

Example 1 – *Solution*

cont'd

Solving for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x$$

Subtracting 2*x*

$$\frac{dy}{dx} = -\frac{x}{y}$$

Canceling the 2′s and dividing by y

Therefore, $\frac{dy}{dx} = -\frac{x}{y}$ when *x* and *y* are related by $x^2 + y^2 = 25$.

Implicit Differentiation

Implicit differentiation involves three steps.

Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides of the equation *with respect to* x. When differentiating a y, include $\frac{dy}{dx}$.
2. Collect all terms involving $\frac{dy}{dx}$ on one side, and all others on the other side.
3. Factor out the $\frac{dy}{dx}$ and solve for it by dividing.

Example 4 – *FINDING AND EVALUATING AN IMPLICIT DERIVATIVE*

For $y^4 + x^4 - 2x^2y^2 = 9$

a. find $\dfrac{dy}{dx}$

b. evaluate it at $x = 2$, $y = 1$

Solution:

$$4y^3\frac{dy}{dx} + 4x^3 - 4xy^2 - 4x^2y\frac{dy}{dx} = 0$$

Differentiating with respect to x, putting constants first

$$4y^3\frac{dy}{dx} - 4x^2y\frac{dy}{dx} = -4x^3 + 4xy^2$$

Collecting dy/dx terms on the left, others on the right

$$(4y^3 - 4x^2y)\frac{dy}{dx} = -4x^3 + 4xy^2$$

Factoring out $\dfrac{dy}{dx}$

Example 1 – *Solution*

cont'd

$$\frac{dy}{dx} = \frac{-4x^3 + 4xy^2}{4y^3 - 4x^2y}$$

← Answer for part (a)

Dividing by $4y^3 - 4x^2y$ to solve for dy/dx

$$= \frac{-x^3 + xy^2}{y^3 - x^2y}$$

Dividing by 4

$$\frac{dy}{dx} = \frac{-(2)^3 + (2)(1)^2}{(1)^3 - (2)^2(1)} = \frac{-6}{-3} = 2$$

Evaluating at $x = 2, y = 1$ gives the answer for part (b)

Implicit Differentiation

Note that in the example above the given point *is* on the curve, since $x = 2$ and $y = 1$ satisfy the original equation:

$$1^4 + 2^4 - 2 \cdot 2^2 \cdot 1 = 1 + 16 - 8 = 9$$

In economics, a **demand equation** is the relationship between the price p of an item and the quantity x that consumers will demand at that price.

Related Rates

Related Rates

Sometimes *both* variables in an equation will be functions of a *third* variable, usually *t* for time.

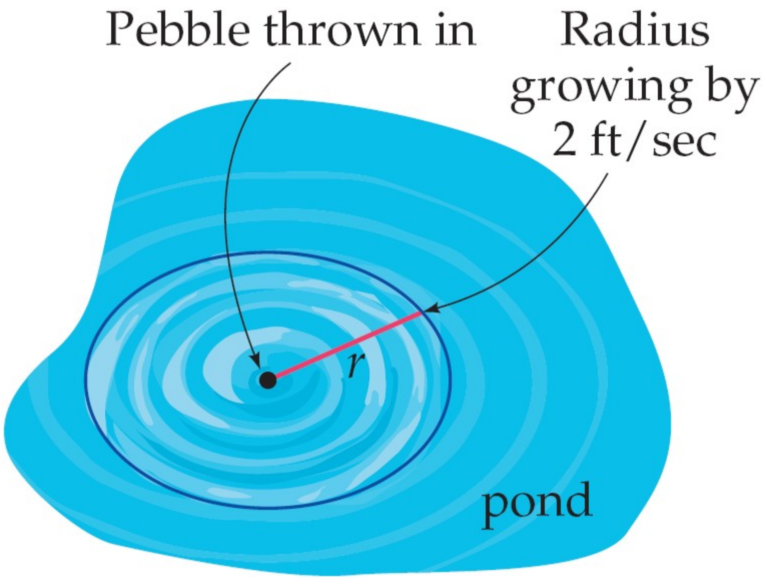
For example, for a seasonal product such as winter coats, the price *p* and weekly sales *x* will be related by a demand equation, and both price *p* and quantity *x* will depend on the time of year.

Differentiating both sides of the demand equation with respect to time *t* will give an equation relating the derivatives dp/dt and dx/dt.

Such “related rates” equations show how fast one quantity is changing relative to another. First, an “everyday” example.

Example 6 – *FINDING RELATED RATES*

A pebble thrown into a pond causes circular ripples to radiate outward. If the radius of the outer ripple is growing by 2 feet per second, how fast is the area of its circle growing at the moment when the radius is 10 feet?



Example 6 – *Solution*

The formula for the area of a circle is $A = \pi r^2$.

Both the area A and the radius r of the circle increase with time, so both are functions of t.

We are told that the radius is increasing by 2 feet per second ($dr/dt = 2$), and we want to know how fast the area is changing (dA/dt).

Example 6 – *Solution*

cont'd

To find the relationship between *dA*/*dt* and *dr*/*dt*, we differentiate both sides of $A = \pi r^2$ with respect to *t*.

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

From $A = \pi r^2$, writing the 2 before the π

$$\frac{dA}{dt} = 2\pi \cdot \underbrace{10}_{r} \cdot \underbrace{2}_{\frac{dr}{dt}}$$

Substituting $r = 10$ and $\frac{dr}{dt} = 2$

$$= 40\pi \approx \underbrace{125.6}_{\substack{\text{Using} \\ \pi \approx 3.14}}$$

Therefore, at the moment when the radius is 10 feet, the area of the circle is growing at the rate of about 126 square feet per second.

Related Rates

To Solve a Related Rate Problem

1. Determine the quantities that are changing with time.
2. Find an equation that relates these quantities (a diagram may be helpful).
3. Differentiate both sides of this equation implicitly with respect to t.
4. Substitute into the new equation any given values for the variables and for the derivatives (interpreted as rates of change).
5. Solve for the remaining derivative and interpret the answer as a rate of change.