

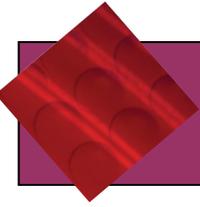
3

Further Applications of Derivatives



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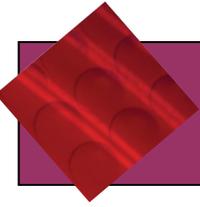
3.5 OPTIMIZING LOT SIZE AND HARVEST SIZE



Introduction

Introduction

- In this section we discuss two important applications of optimization, one economic and one ecological.
- The first concerns the most efficient way for a business to order merchandise (or for a manufacturer to produce merchandise), and the second concerns the preservation of animal populations that are harvested by people.
- Either of these applications may be read independently of the other.



Minimizing Inventory Costs

Minimizing Inventory Costs

A business encounters two kinds of costs in maintaining inventory: **storage costs** (warehouse and insurance costs for merchandise not yet sold) and **reorder costs** (delivery and bookkeeping costs for each order).

For example, if a furniture store expects to sell 250 sofas in a year, it could order all 250 at once (incurring high storage costs), or it could order them in many small lots, say 50 orders of five each, spaced throughout the year (incurring high reorder costs).

Obviously, the best order size (or **lot size**) is the one that minimizes the total of storage plus reorder costs.

Example 1 – MINIMIZING INVENTORY COSTS

A furniture showroom expects to sell 250 sofas a year. Each sofa costs the store \$300, and there is a fixed charge of \$500 per order. If it costs \$100 to store a sofa for a year, how large should each order be and how often should orders be placed to minimize inventory costs?

Solution:

Let

x = lot size

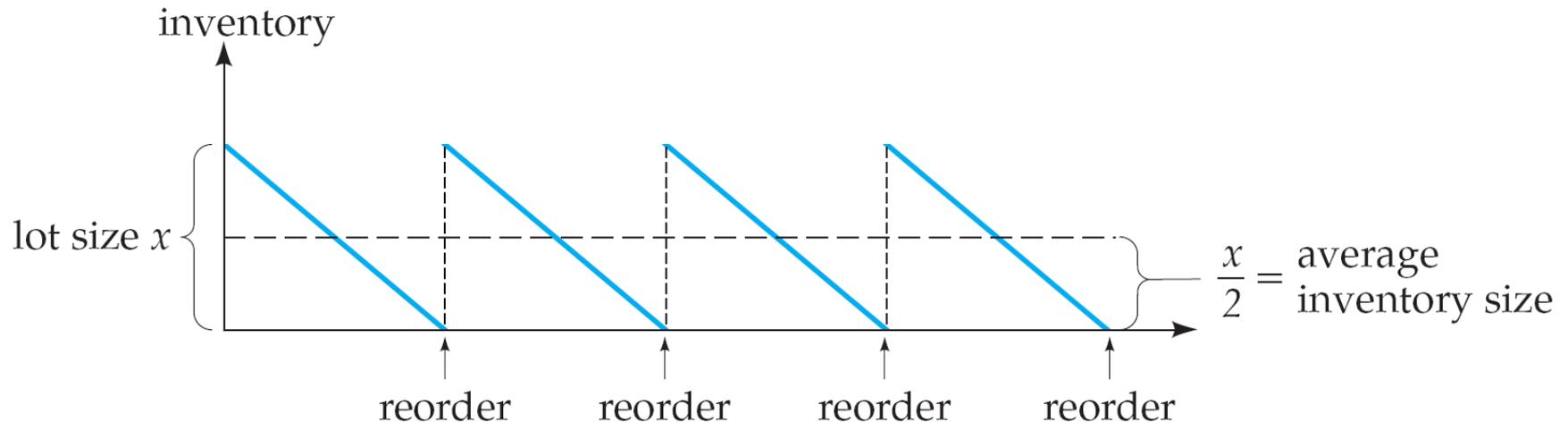
Number of sofas in each order

Example 1 – Solution

cont'd

Storage Costs:

If the sofas sell steadily throughout the year, and if the store reorders x more whenever the stock runs out, then its inventory during the year looks like the following graph.



Example 1 – Solution

cont'd

Notice that the inventory level varies from the lot size x down to zero, with an average inventory of $x/2$ sofas throughout the year.

Because it costs \$100 to store a sofa for a year, the total (annual) storage costs are

$$\begin{aligned} \left(\begin{array}{c} \text{Storage} \\ \text{costs} \end{array} \right) &= \left(\begin{array}{c} \text{Storage} \\ \text{per item} \end{array} \right) \cdot \left(\begin{array}{c} \text{Average num-} \\ \text{ber of items} \end{array} \right) \\ &= 100 \cdot \frac{x}{2} \\ &= 50x \end{aligned}$$

Example 1 – Solution

cont'd

Reorder Costs:

Each sofa costs \$300, so an order of lot size x costs $300x$, plus the fixed order charge of \$500:

$$\left(\begin{array}{c} \text{Cost} \\ \text{per order} \end{array} \right) = 300x + 500$$

The yearly supply of 250 sofas, with x sofas in each order, requires $\frac{250}{x}$ orders. (For example, 250 sofas at 5 per order require $\frac{250}{5} = 50$ orders.)

Therefore, the yearly reorder costs are

$$\left(\begin{array}{c} \text{Reorder} \\ \text{costs} \end{array} \right) = \left(\begin{array}{c} \text{Cost} \\ \text{per order} \end{array} \right) \cdot \left(\begin{array}{c} \text{Number} \\ \text{of orders} \end{array} \right) = (300x + 500) \cdot \left(\frac{250}{x} \right)$$

Example 1 – Solution

cont'd

Total Cost:

$C(x)$ is storage costs plus reorder costs:

$$\begin{aligned} C(x) &= \left(\begin{array}{c} \text{Storage} \\ \text{costs} \end{array} \right) + \left(\begin{array}{c} \text{Reorder} \\ \text{costs} \end{array} \right) \\ &= 100 \frac{x}{2} + (300x + 500) \left(\frac{250}{x} \right) \\ &= 50x + 75,000 + 125,000x^{-1} \end{aligned}$$

Using the storage and reorder costs found earlier

Simplifying

To minimize $C(x)$, we differentiate:

$$\begin{aligned} C'(x) &= 50 - 125,000x^{-2} \\ &= 50 - \frac{125,000}{x^2} \end{aligned}$$

Differentiating

$$C = 50x + 75,000 + 125,000x^{-1}$$

Example 1 – Solution

cont'd

$$50 - \frac{125,000}{x^2} = 0$$

Setting the derivative equal to zero

$$50x^2 = 125,000$$

Multiplying by x^2 and adding 125,000 to each side

$$x^2 = \frac{125,000}{50} = 2500$$

Dividing each x^2 side by 50

$$x = 50$$

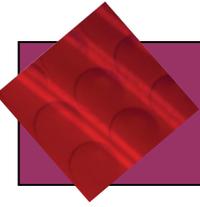
Taking square roots ($x > 0$) gives lot size 50

$$C''(x) = 250,000x^{-3} = 250,000 \frac{1}{x^3}$$

C'' is positive, so C is minimized at $x = 50$

At 50 sofas per order, the yearly 250 will require $\frac{250}{50} = 5$ orders.

Therefore: Lot size is 50 sofas, with orders placed five times a year.



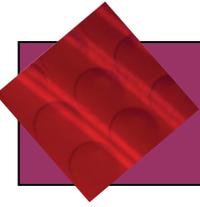
Modifications and Assumptions

Modifications and Assumptions

If the number of orders per year is not a whole number, say 7.5 orders per year, we just interpret it as 15 orders in 2 years, and handle it accordingly.

We made two major assumptions in Example 1. We assumed that there was a steady demand, and that orders were equally spaced throughout the year.

These are reasonable assumptions for many products, while for seasonal products such as bathing suits or winter coats, separate calculations can be done for the “on” and “off” seasons.



Production Runs

Production Runs

Similar analysis applies to manufacturing.

For example, if a book publisher can estimate the yearly demand for a book, she may print the yearly total all at once, incurring high storage costs, or she may print them in several smaller runs throughout the year, incurring setup costs for each run.

Here the setup costs for each printing run play the role of the reorder costs for a store.

Example 2 – *MINIMIZING INVENTORY COSTS FOR A PUBLISHER*

A publisher estimates the annual demand for a book to be 4000 copies. Each book costs \$8 to print, and setup costs are \$1000 for each printing. If storage costs are \$2 per book per year, find how many books should be printed per run and how many printings will be needed if costs are to be minimized.

Solution:

Let

x = the number of books in each run

Example 2 – Solution

cont'd

Storage Costs

As in Example 1, an average of $\frac{x}{2}$ books are stored throughout the year, at a cost of \$2 each, so annual storage costs are

$$\left(\begin{array}{c} \text{Storage} \\ \text{costs} \end{array} \right) = \left(\frac{x}{2} \right) \cdot 2 = x$$

Production Costs

The cost per run is

$$\left(\begin{array}{c} \text{Costs} \\ \text{per run} \end{array} \right) = 8x + 1000$$

x books at \$8 each, plus
\$1000 setup costs

Example 2 – Solution

cont'd

The 4000 books at x books per run will require $\frac{4000}{x}$ runs.
Therefore, production costs are

$$\left(\begin{array}{c} \text{Production} \\ \text{costs} \end{array} \right) = (8x + 1000) \left(\frac{4000}{x} \right) \quad \text{Cost per run times number of runs}$$

Total Cost

The total cost is storage costs plus production costs:

$$\begin{aligned} C(x) &= x + (8x + 1000) \left(\frac{4000}{x} \right) && \text{Storage + production} \\ &= x + 32,000 + 4,000,000x^{-1} && \text{Multiplying out} \end{aligned}$$

Example 2 – *Solution*

cont'd

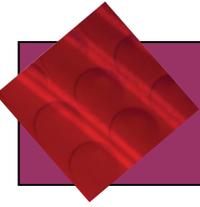
We differentiate, set the derivative equal to zero, and solve, just as before (omitting the details), obtaining $x = 2000$.

The second-derivative test will show that costs are minimized at 2000 books per run.

The 4000 books require $\frac{4000}{2000} = 2$ printings.

Therefore, the publisher should:

Print 2000 books per run, with two printings.



Maximum Sustainable Yield

Maximum Sustainable Yield

The next application involves industries such as fishing, in which a naturally occurring animal population is “harvested.”

Harvesting some of the population means more food and other resources for the remaining population so that it will expand and replace the harvested portion.

But taking too large a harvest will kill off the animal population.

We want to find the **maximum sustainable yield**, the largest amount that may be harvested year after year and still have the population return to its previous level the following year.

Maximum Sustainable Yield

For some animals one can determine a **reproduction function** $f(p)$ which gives the expected population a year from now if the present population is p .

Reproduction Function

A reproduction function $f(p)$ gives the population a year from now if the current population is p .

Maximum Sustainable Yield

Suppose that we have a reproduction function f and a current population of size p , which will therefore grow to size $f(p)$ next year. The *amount of growth* in the population during that year is

$$\left(\begin{array}{l} \text{Amount} \\ \text{of growth} \end{array} \right) = f(p) - p$$

Next year's population

Current population

Harvesting this amount removes only the *growth*, returning the population to its former size p .

Maximum Sustainable Yield

The population will then repeat this growth, and taking the same harvest $f(p) - p$ will cause this situation to repeat itself year after year. The quantity $f(p) - p$ is called the sustainable yield.

Sustainable Yield

For reproduction function $f(p)$, the sustainable yield is

$$Y(p) = f(p) - p$$

Maximum Sustainable Yield

We want the population size p that maximizes the sustainable yield $Y(p)$. To maximize $Y(p)$, we set its derivative equal to zero:

$$Y'(p) = f'(p) - 1 = 0$$

Derivative of $Y = f(p) - p$

$$f'(p) = 1$$

Solving for $f'(p)$

For a given reproduction function $f(p)$, we find the maximum sustainable yield by solving this equation (provided that the second-derivative test gives $Y''(p) = f''(p) < 0$).

Maximum Sustainable Yield

Maximum Sustainable Yield

For reproduction function $f(p)$, the population p that results in the maximum sustainable yield is the solution to

$$f'(p) = 1$$

(provided that $f''(p) < 0$). The maximum sustainable yield is then

$$Y(p) = f(p) - p$$

Once we calculate the population p that gives the maximum sustainable yield, we wait until the population reaches this size and then harvest, year after year, an amount $Y(p)$.

Example 3 – FINDING MAXIMUM SUSTAINABLE YIELD

The reproduction function for the American lobster in an East Coast fishing area is $f(p) = -0.02p^2 + 2p$ (where p and $f(p)$ are in thousands). Find the population p that gives the maximum sustainable yield and find the size of the yield.

Solution:

We set the derivative of the reproduction function equal to 1:

$$\begin{aligned} f'(p) &= -0.04p + 2 = 1 && \text{Differentiating} \\ -0.04p &= -1 && \text{Subtracting 2 from each side} \\ p &= \frac{-1}{-0.04} = 25 && \text{Dividing by 0.04} \end{aligned}$$

Example 3 – Solution

cont'd

The second derivative is $f''(p) = -0.04$, which is negative, showing that $p = 25$ (thousand) is the population that gives the maximum sustainable yield.

The actual yield is found from the yield function

$f(p) = Y(p) - p$:

$$\begin{aligned} Y(p) &= -0.02p^2 + 2p - p \\ &= \underbrace{0.02p^2 + p}_{f(p)} \end{aligned}$$

$$Y(25) = -0.02(25)^2 + 25$$

Evaluating at $p = 25$ (thousand)

The population size for the maximum sustainable yield is 25,000, and the yield is 12,500 lobsters. 25,000 from $p = 25$