3 Further Applications of Derivatives

3.2 GRAPHING USING THE FIRST AND SECOND DERIVATIVES

Introduction

Introduction

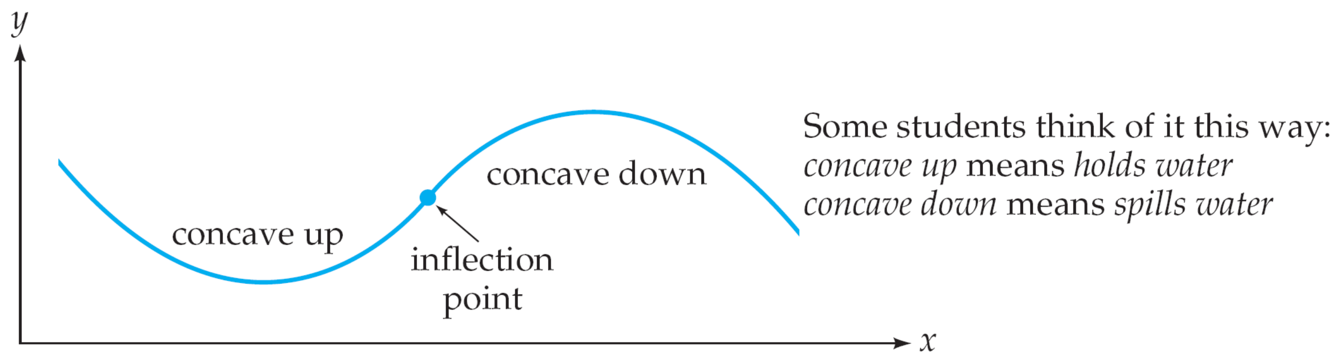
- In this section we will use the *second* derivative to find the **concavity** or **curl** of the curve, and to define the important concept of **inflection point.**

- The second derivative also gives us a very useful way to distinguish between maximum and minimum points of a curve.

Concavity and Inflection Points

Concavity and Inflection Points

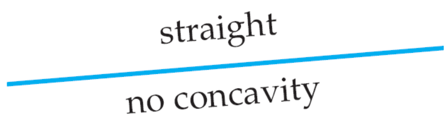
- A curve that curls upward is said to be **concave up,** and a curve that curls downward is said to be **concave down.**
- A point where the concavity *changes* (from up to down or down to up) is called an **inflection point.**



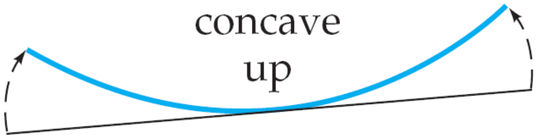
Concavity shows how a curve *curls* or *bends* away from straightness.

Concavity and Inflection Points

A straight line (with any slope) has *no concavity.*



However, bending the two ends *upward* makes it *concave up,*



and bending the two ends *downward* makes it *concave down.*

As these pictures show, a curve that is concave *up* lies *above* its tangent, while a curve that is concave *down* lies *below* its tangent (except at the point of tangency).

Concavity and Inflection Points

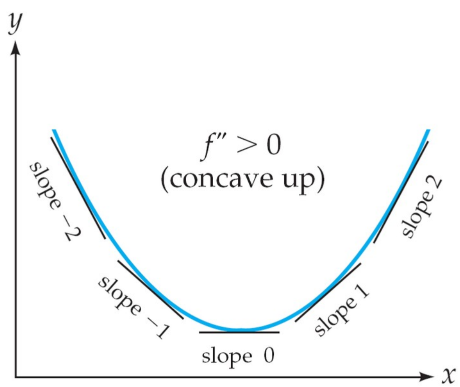
Concavity and Inflection Points

On an interval:

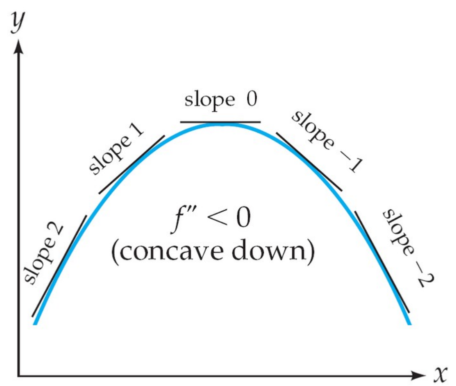
$f'' > 0$ means that f is *concave up* (curls upward).

$f'' < 0$ means that f is *concave down* (curls downward).

An *inflection point* is where the concavity *changes* (f'' must be zero or undefined).



$f'' > 0$ means that the slope is increasing, so f is *concave up*.



$f'' < 0$ means that the slope is decreasing, so f is *concave down*.

EXAMPLE 1 – *GRAPHING AND INTERPREATING A COMPANY'S ANNUAL PROFIT FUNCTION*

A company's annual profit after x years is

$f(x) = x^3 - 9x^2 + 24x$ million dollars (for $x \geq 0$). Graph this function, showing all relative extreme points and inflection points. Interpret the inflection points.

Solution :

$f'(x) = 3x^2 - 18x + 24$ Differentiating

$= 3(x^2 - 6x + 8)$

$= 3(x - 2)(x - 4)$ Factoring

Example 1 – *Solution*

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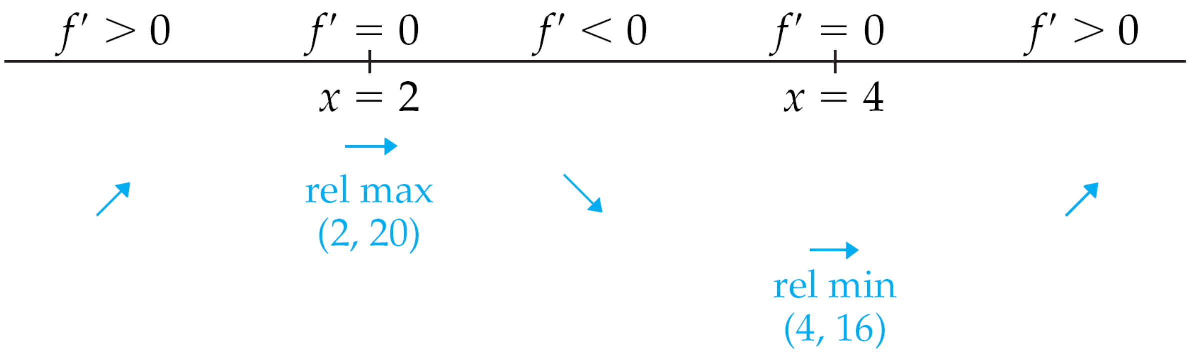
The critical numbers are $x = 2$ and $x = 4$, and the sign diagram for f' (found in the usual way) is



To find the inflection points, we calculate the second derivative:

$f''(x) = 6x - 18 = 6(x - 3)$ Differentiating $f'(x) = 3x^2 - 18x + 24$

Example 1 – *Solution*

cont'd

This is zero at $x = 3$, which we enter on a sign diagram for the *second* derivative.

$f'' = 0$

$x = 3$

← Behavior of f''

← Where f'' is zero or undefined

We use test points to determine the sign of $f''(x) = 6(x - 3)$ on either side of 3, just as we did for the first derivative.

$f''(2) = 6(2 - 3) < 0$ $f''(4) = 6(4 - 3) > 0$

$f'' < 0$ $f'' = 0$ $f'' > 0$

$x = 3$

con dn   con up

IP (3, 18)

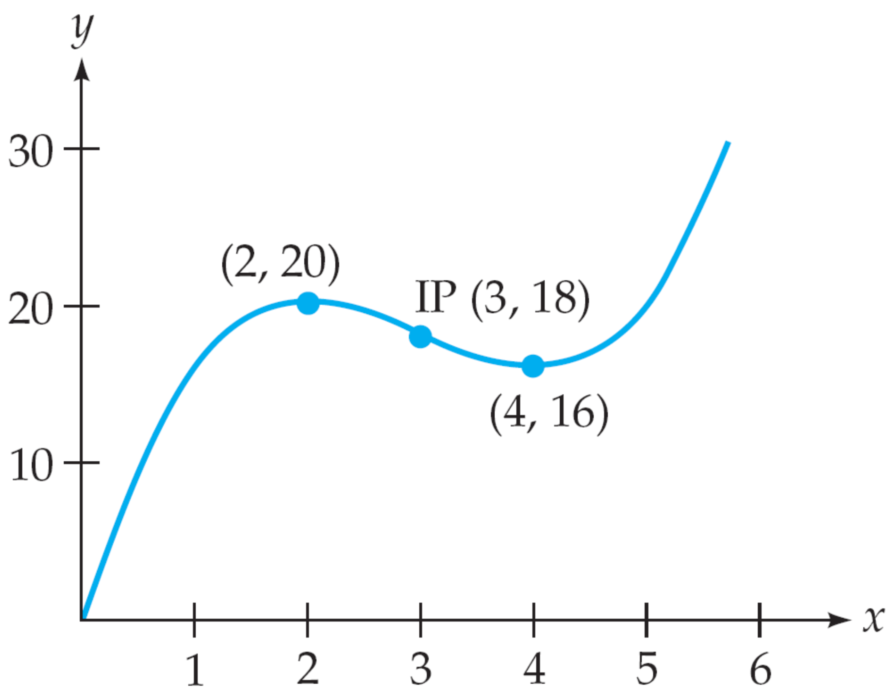
Concave down, concave up (so concavity *does* change)

IP means *inflection point*. The 18 comes from substituting $x = 3$ into $f(x) = x^3 - 9x^2 + 24x$

Example 1 – *Solution*

cont'd

By hand, we would plot the relative maximum (), minimum (), and inflection point and sketch the curve according to the sign diagrams, being sure to show the concavity changing at the inflection point.



Example 1 – *Solution*

cont'd

Interpretation of the inflection point:

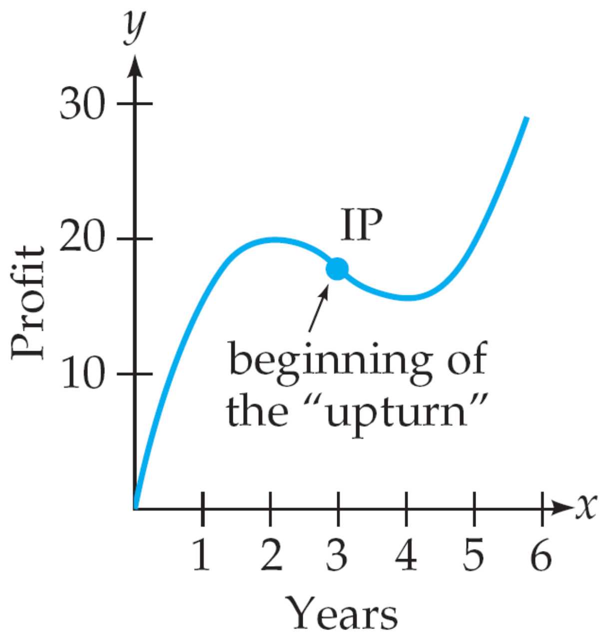
Observe what the graph shows—that the company's profit increased (up to year 2), then decreased (up to year 4),and then increased again.

The inflection point at $x = 3$ is where the profit *first began to show signs of improvement.*

Example 1 – *Solution*

cont'd

It marks the end of the period of increasingly steep decline and the first sign of an "upturn," where a clever investor might begin to "buy in." See the graph below.



Inflection Points in the Real World

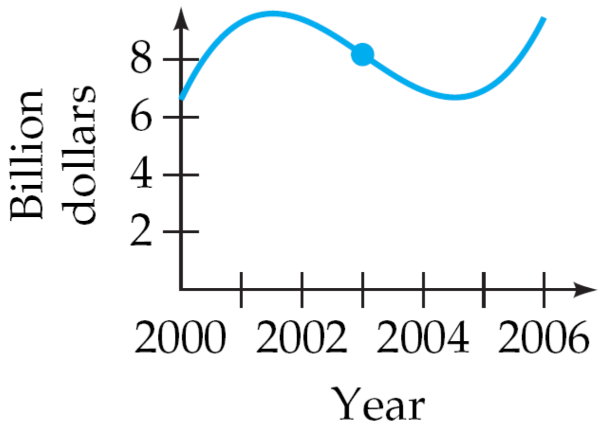
Inflection Points in the Real World

Inflection points occur in many everyday situations.

The function in Example 1, while constructed for ease of calculation, is essentially the graph of net income for AT&T over recent years, shown below.

The inflection point represents the first sign of the upturn in AT&T's net income, which occurred in 2003.

Source: Standard & Poor's



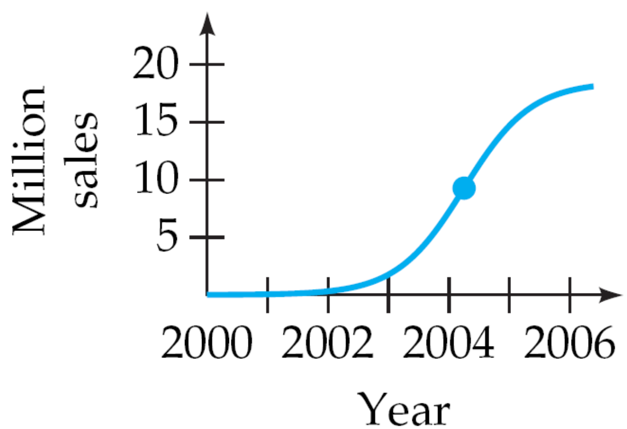
Inflection Points in the Real World

The graph below shows the annual sales of portable MP3 players.

The inflection point, occurring between 2004 and 2005, marks the end of the period of increasingly rapid sales growth and first sign of approaching market saturation.

This is when savvy manufacturers might begin to curtail new investment in MP3 production.

Source: Consumer USA 2008

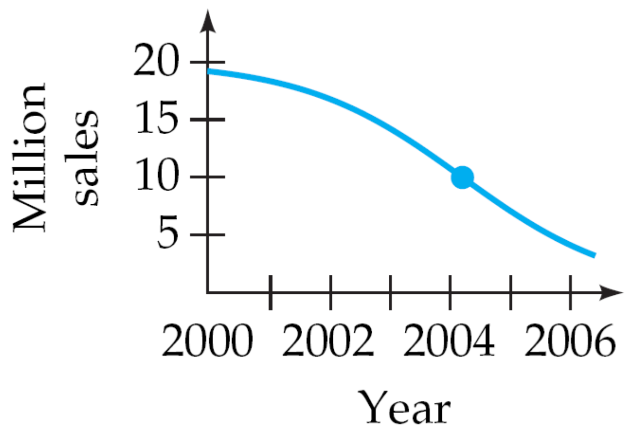


Inflection Points in the Real World

The graph below shows the annual sales of analog cameras a few years after the introduction of digital cameras.

The inflection point, occurring in 2004, marks the end of the increasingly steep sales decline and the first sign of steadying but lower sales.

Source: Euromonitor

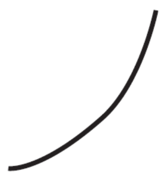


Inflection Points in the Real World

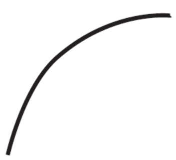
Distinguish carefully between slope and concavity: *slope* measures *steepness,* whereas *concavity* measures *curl.* All combinations of slope and concavity are possible.

A graph may be

Increasing and concave *up*
($f' > 0$, $f'' > 0$), such as



Increasing and concave *down*
($f' > 0$, $f'' < 0$), such as

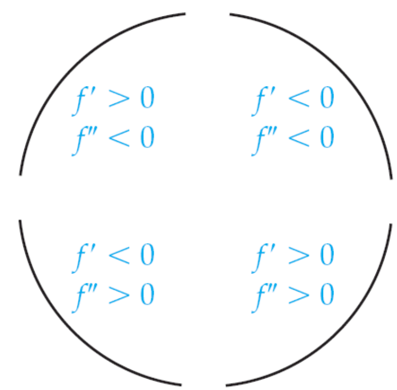


Inflection Points in the Real World

Decreasing and concave *up* ($f' < 0$, $f'' > 0$), such as

Decreasing and concave *down* ($f' < 0$, $f'' < 0$), such as

The four quarters of a circle illustrate all four possibilities, as shown at the right.



Example 2 – *GRAPHING A FRACTIONAL POWER FUNCTION*

Graph $f(x) = 18x^{1/3}$.

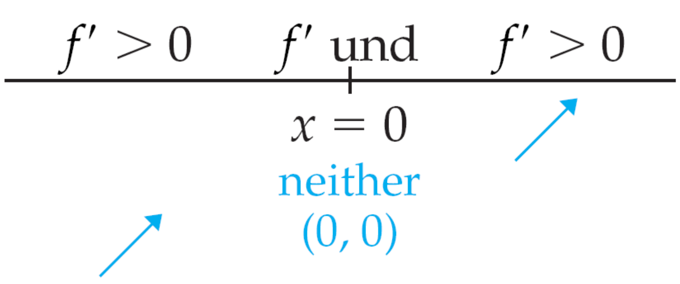
Solution :

The derivative is

$$f'(x) = 6x^{-2/3} = \frac{6}{\sqrt[3]{x^2}}$$

Undefined at $x = 0$
(zero denominator)

The sign diagram for f' is



f' is undefined at $x = 0$ and positive on either side (using test points)

Example 2 – *Solution*

cont'd

The *second* derivative is

$$f''(x) = -4x^{-5/3} = \frac{-4}{\sqrt[3]{x^5}}$$

Also undefined at $x = 0$

The sign diagram for *f''* is

$f'' > 0$	f'' und	$f'' < 0$
con up	$x = 0$	con dn
	IP (0, 0)	

Concavity is different on either side of $x = 0$ (using test points), so there *is* an inflection point at $x = 0$

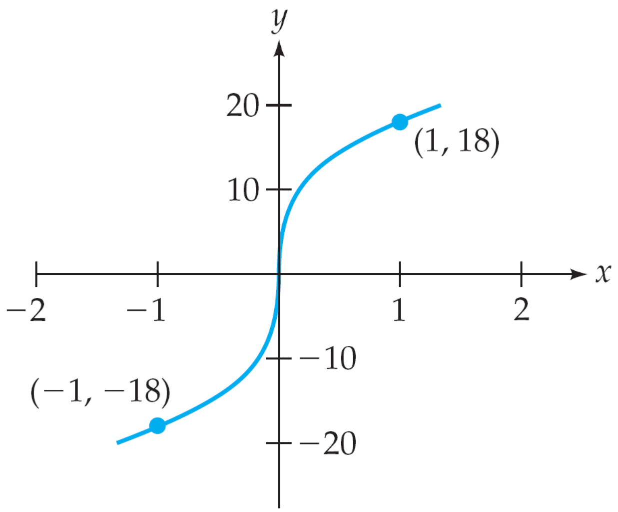
Example 2 – *Solution*

cont'd

Based on this information, we may graph the function by hand:

By hand, we use the sign diagrams to draw the curve to the *left* of $x = 0$ as *increasing* and concave *up,* and to

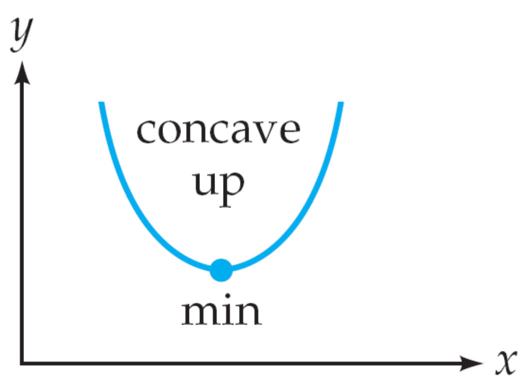
the right of $x = 0$ as increasing and concave *down,* with the two parts meeting at the origin. The scale comes from calculating the points(1, 18) and (–1, –18).



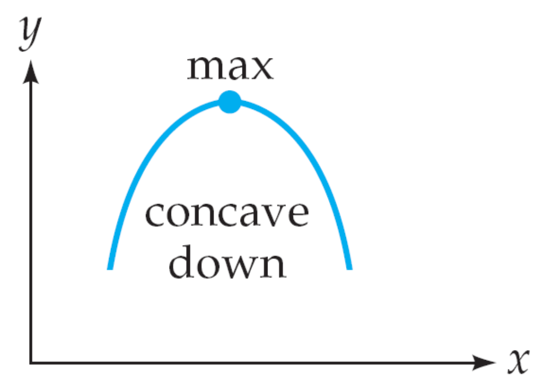
Second-Derivative Test

Second-Derivative Test

Determining whether a twice-differentiable function has a relative maximum or minimum at a critical number is merely a question of concavity: concave *up* means a relative *minimum,* and concave *down* means a relative *maximum.*



Concave *up* at a critical number: relative *minimum*.



Concave *down* at a critical number: relative *maximum*.

Since the second derivative determines concavity, we have the *second-derivative test*.

Second-Derivative Test

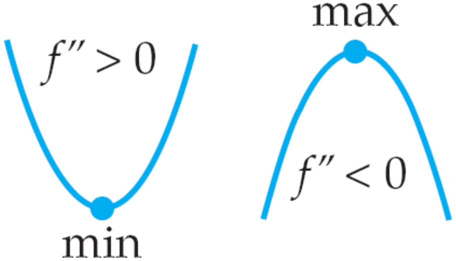
Second-Derivative Test for Relative Extreme Points

If $x = c$ is a critical number of f at which f'' is defined, then

$f''(c) > 0$ means that f has a relative *minimum* at $x = c$.

$f''(c) < 0$ means that f has a relative *maximum* at $x = c$.

To use the second-derivative test, first find all critical numbers, substitute each into the second derivative (if possible), and determine the sign of the result: a *positive* result means a *minimum* at the critical number, and a *negative* result means a *maximum.*



Example 4 – *USING THE SECOND-DERIVATIVE TEST*

Use the second-derivative test to find all relative extreme points of

$f(x) = x^3 – 9x^2 + 24x.$

Solution :

$$f'(x) = 3x^2 - 18x + 24 \qquad \text{The derivative}$$

$$= 3(x^2 - 6x + 8) = 3(x - 2)(x - 4) \qquad \text{Factoring}$$

$$\text{CN} \begin{cases} x = 2 \\ x = 4 \end{cases} \qquad \text{Critical numbers}$$

Example 4 – *Solution*

cont'd

We substitute each critical number into $f''(x) = 6x - 18$.

At $x = 2$: $f''(2) = 6 \cdot 2 - 18 = -6$ (negative) $f''(x) = 6x - 18$ at $x = 2$

Therefore, f has a relative *maximum* at $x = 2$.

At $x = 4$: $f''(4) = 6 \cdot 4 - 18 = 6$ (positive) $f''(x) = 6x - 18$ at $x = 4$

Therefore, f has a relative *minimum* at $x = 4$.

The relative maximum at $x = 2$ and the relative minimum at $x = 4$ are exactly what we found before when we graphed this function as shown on the left.