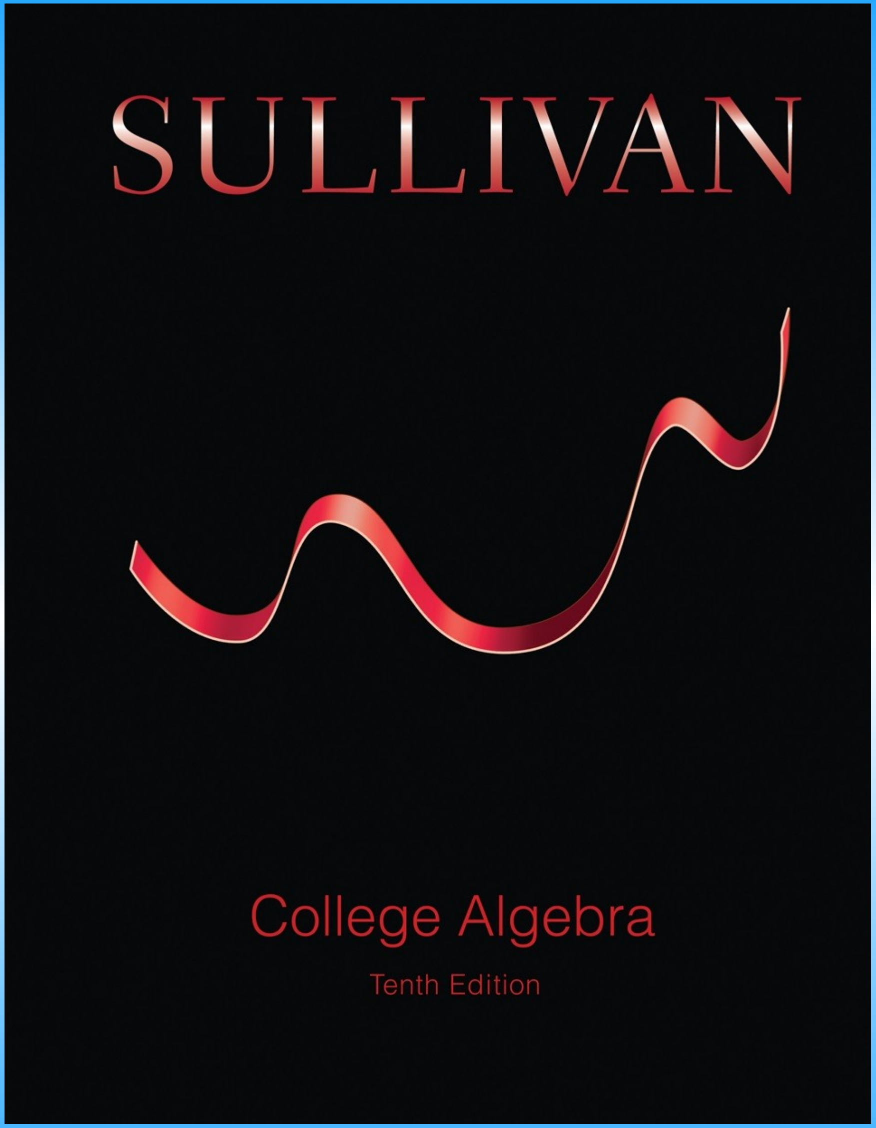
Chapter 1

Section 3



1.3 Complex Numbers; Quadratic Equations in the Complex Number System *

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Classification of Numbers (Section R.1, pp. 4–5)
- Rationalizing Denominators (Section R.8, p. 75)

Now Work the **'Are You Prepared?'** problems on page 111.

OBJECTIVES
1 Add, Subtract, Multiply, and Divide Complex Numbers (p. 105)
2 Solve Quadratic Equations in the Complex Number System (p. 109)

Definition

The **imaginary unit**, which we denote by $\boldsymbol{i}$, is the number whose square is -1. That is,

$$i^2 = -1$$

Definition

Complex numbers are numbers of the form $\boldsymbol{a + bi}$, where a and b are real numbers. The real number a is called the **real part** of the number $a + bi$; the real number b is called the **imaginary part** of $a + bi$; and i is the imaginary unit, so $i^2 = -1$.

Add, Subtract, Multiply, and Divide Complex Numbers

Equality of Complex Numbers

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d \qquad \textbf{(1)}$$

Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad \textbf{(2)}$$

Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i \qquad \textbf{(3)}$$

Example

Adding and Subtracting Complex Numbers

(a) $(3 + 5i) + (-2 + 3i) = [3 + (-2)] + (5 + 3)i = 1 + 8i$

(b) $(6 + 4i) - (3 + 6i) = (6 - 3) + (4 - 6)i = 3 + (-2)i = 3 - 2i$

Example

Multiplying Complex Numbers

$$(5+3i)\cdot(2+7i) \underset{\substack{\uparrow \\ \text{Distributive Property}}}{=} 5\cdot(2+7i)+3i(2+7i) \underset{\substack{\uparrow \\ \text{Distributive Property}}}{=} 10+35i+6i+21i^2$$

$$\underset{\substack{\uparrow \\ i^2 = -1}}{=} 10+41i+21(-1)$$

$$= -11+41i$$

Product of Complex Numbers

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i \qquad \textbf{(4)}$$

Definition

If $z = a + bi$ is a complex number, then its **conjugate**, denoted by $\overline{z}$, is defined as

$$\boxed{\overline{z} = \overline{a + bi} = a - bi}$$

Example

Multiplying a Complex Number by Its Conjugate

Find the product of the complex number $z = 3 + 4i$ and its conjugate $\overline{z}$.

Solution

Since $\overline{z} = 3 - 4i$, we have

$$z\overline{z} = (3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25$$

Theorem

The product of a complex number and its conjugate is a nonnegative real number. That is, if $z = a + bi$, then

$$z\bar{z} = a^2 + b^2 \qquad \textbf{(5)}$$

Example

Writing the Quotient of Two Complex Numbers in Standard Form

Write each of the following in standard form.

(a) $\dfrac{1 + 4i}{5 - 12i}$ (b) $\dfrac{2 - 3i}{4 - 3i}$

Solution

(a) $$\frac{1+4i}{5-12i} = \frac{1+4i}{5-12i}\cdot\frac{5+12i}{5+12i} = \frac{5+12i+20i+48i^2}{25+144}$$

$$= \frac{-43+32i}{169} = -\frac{43}{169} + \frac{32}{169}i$$

(b) $$\frac{2-3i}{4-3i} = \frac{2-3i}{4-3i}\cdot\frac{4+3i}{4+3i} = \frac{8+6i-12i-9i^2}{16+9} = \frac{17-6i}{25} = \frac{17}{25} - \frac{6}{25}i$$

Theorem

The conjugate of a real number is the real number itself.

Theorem

The conjugate of the conjugate of a complex number is the complex number itself.

$$\left(\overline{\overline{z}}\right) = z \tag{6}$$

The conjugate of the sum of two complex numbers equals the sum of their conjugates.

$$\overline{z + w} = \overline{z} + \overline{w} \tag{7}$$

The conjugate of the product of two complex numbers equals the product of their conjugates.

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \tag{8}$$

Powers of *i*

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^6 = i^4 \cdot i^2 = -1$$
$$i^7 = i^4 \cdot i^3 = -i$$
$$i^8 = i^4 \cdot i^4 = 1$$

Example

Evaluating Powers of *i*

(a) $i^{27} = i^{24} \cdot i^{3} = \left(i^{4}\right)^{6} \cdot i^{3} = 1^{6} \cdot i^{3} = -i$

(b) $i^{101} = i^{100} \cdot i^{1} = \left(i^{4}\right)^{25} \cdot i = 1^{25} \cdot i = i$

Solve Quadratic Equations in the Complex Number System

Definition

If N is a positive real number, we define the **principal square root of $-N$**, denoted by $\sqrt{-N}$, as

$$\boxed{\sqrt{-N} = \sqrt{N}\,i}$$

where i is the imaginary unit and $i^2 = -1$.

Example

Solving Equations

Solve each equation in the complex number system.

(a) $x^2 = 4$ (b) $x^2 = -9$

Solution

(a) $x^2 = 4$

$$x = \pm\sqrt{4} = \pm 2$$

The equation has two solutions, -2 and 2. The solution set is $\{-2, 2\}$.

(b) $x^2 = -9$

$$x = \pm\sqrt{-9} = \pm\sqrt{9}\,i = \pm 3i$$

The equation has two solutions, $-3i$ and $3i$. The solution set is $\{-3i, 3i\}$.

Theorem

Quadratic Formula

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(9)}$$

Example

Solving a Quadratic Equation in the Complex Number System

Solve the equation $x^2 - 4x + 8 = 0$ in the complex number system.

Solution

Here $a = 1, b = -4, c = 8,$ and $b^2 - 4ac = (-4)^2 - 4(1)(8) = -16.$ Using equation (9), we find that

$$x = \frac{-(-4) \pm \sqrt{-16}}{2(1)} = \frac{4 \pm \sqrt{16}\,i}{2} = \frac{4 \pm 4i}{2} = \frac{\cancel{2}(2 \pm 2i)}{\cancel{2}} = 2 \pm 2i$$

The equation has two solutions: $2 - 2i$ and $2 + 2i$.

The solution set is $\{2 - 2i, 2 + 2i\}$.

Check: $2 + 2i$: $\quad (2 + 2i)^2 - 4(2 + 2i) + 8 = 4 + \cancel{8i} + 4i^2 - \cancel{8} - \cancel{8i} + \cancel{8}$

$$= 4 + 4i^2$$

$$= 4 - 4 = 0$$

$2 - 2i$: $\quad (2 - 2i)^2 - 4(2 - 2i) + 8 = 4 - \cancel{8i} + 4i^2 - \cancel{8} + \cancel{8i} + \cancel{8}$

$$= 4 - 4 = 0$$

The Discriminant

The discriminant of a quadratic equation still serves as a way to determine the character of the solutions.

Character of the Solutions of a Quadratic Equation

In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients.

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.
3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

Example

Determining the Character of the Solutions of a Quadratic Equation

Without solving, determine the character of the solutions of each equation.

(a) $3x^2 + 4x + 5 = 0$ (b) $2x^2 + 4x + 1 = 0$ (c) $9x^2 - 6x + 1 = 0$

Solution

(a) Here $a = 3, b = 4,$ and $c = 5,$ so $b^2 - 4ac = 16 - 4(3)(5) = -44.$ The solutions are two complex numbers that are not real and are conjugates of each other.

(b) Here $a = 2, b = 4,$ and $c = 1,$ so $b^2 - 4ac = 16 - 8 = 8.$ The solutions are two unequal real numbers.

(c) Here $a = 9, b = -6,$ and $c = 1,$ so $b^2 - 4ac = 36 - 4(9)(1) = 0.$ The solution is a repeated real number—that is, a double root.