2 Derivatives and Their Uses

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2.7 NONDIFFERENTIABLE FUNCTIONS

Introduction

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- In spite of all of the rules of differentiation, there are **nondifferentiable functions**—functions that cannot be differentiated at certain values.
- We begin this section by exhibiting such a function (the absolute value function) and showing that it is not differentiable at $x = 0$.
- We will then discuss general geometric conditions for a function to be nondifferentiable.
- Knowing where a function is not differentiable is important for understanding graphs and for interpreting answers from a graphing calculator.

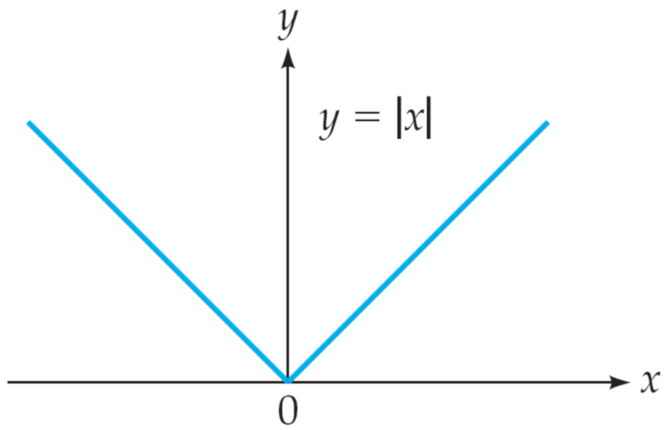
Absolute Value Function

Absolute Value Function

We know that the absolute value function,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Although the absolute value function is *defined* for *all* values of *x*, we will show that it is *not* differentiable at *x* = 0.



The graph of the absolute value function $f(x) = |x|$ has a "corner" at the origin.

Example 1 – *SHOWING NONDIFFERENTIABILITY*

Show that $f(x) = |\,x\,|$ is not differentiable at $x = 0$.

Solution:

We have no "rules" for differentiating the absolute value function, so we must use the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists.

It is this provision, which until now we have steadfastly ignored, that will be important in this example.

Example 1 – *Solution*

cont'd

We will show that this limit, and hence the derivative, does not exist at x = 0.

For x = 0 the definition becomes

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \underbrace{\frac{|h| - |0|}{h}}_{\text{Using } f(x) = |x|} = \lim_{h \to 0} \frac{|h|}{h}$$

Example 1 – *Solution*

cont'd

For this limit to exist, the two one-sided limits must exist and have the same value.

The limit from the right is

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

Since $|h| = h$ for $h > 0$

$\frac{h}{h} = 1$

Example 1 – *Solution*

cont'd

The limit from the left is:

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

For $h < 0, |h| = -h$ (the negative sign makes the negative h positive)

$$\frac{-h}{h} = -1$$

Since the one-sided limits do not agree (one is +1 and the other is –1), the limit $\lim_{h \to 0} \frac{|h|}{h}$ does not exist, so *the derivative does not exist.*

This is what we wanted to show—that the absolute value function is not differentiable at $x = 0$.

Geometric Explanation of Nondifferentiability

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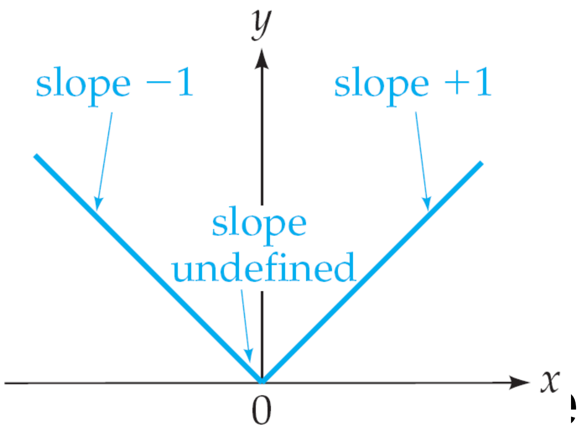
We can give a geometric and intuitive reason why the absolute value function is not differentiable at $x = 0$.

Its graph consists of two straight lines with slopes +1 and –1 that meet in a corner at the origin.

To the right of the origin the slope is +1 and to the left of the origin the slope is –1, but *at* the origi
the two conflicting slope
it impossible to

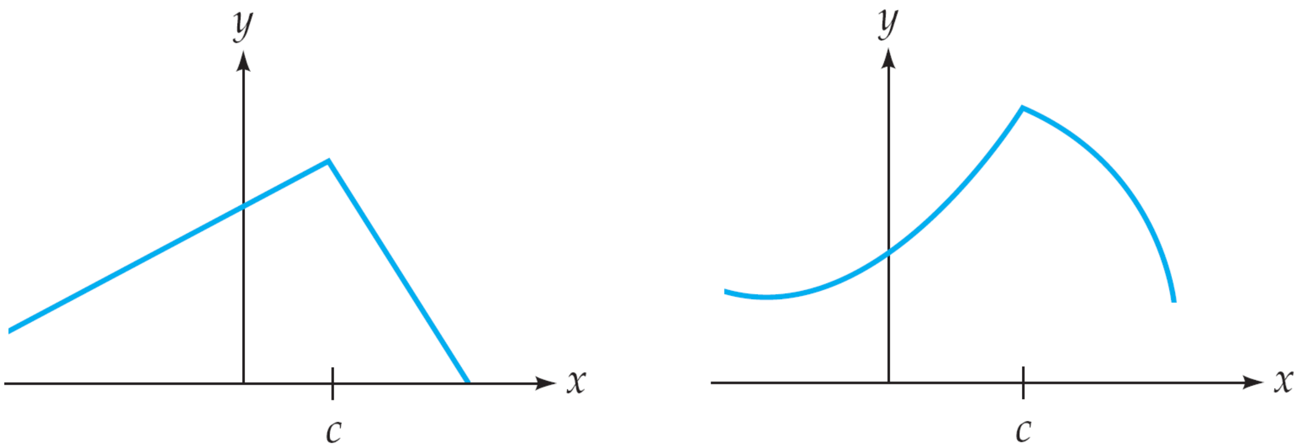


Therefore, the slope (and hence the derivative) is undefined at $x = 0$.

Other Nondifferentiable Functions

Other Nondifferentiable Functions

For the geometric and intuitive reason, at any **corner point** of a graph, where two different slopes conflict, the function will not be differentiable.

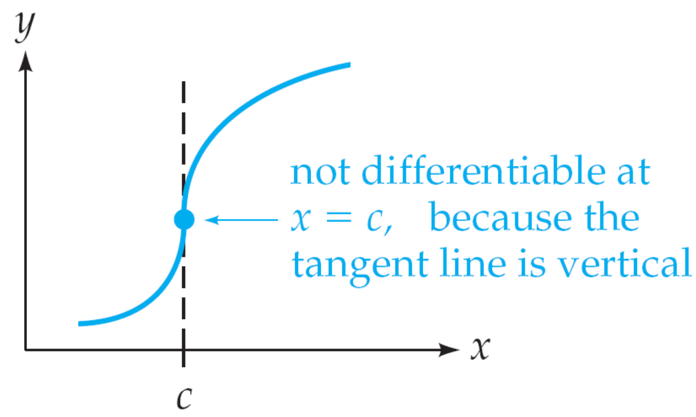


Each of the functions graphed here has a "corner point" at $x = c$, and so is not differentiable at $x = c$.

Other Nondifferentiable Functions

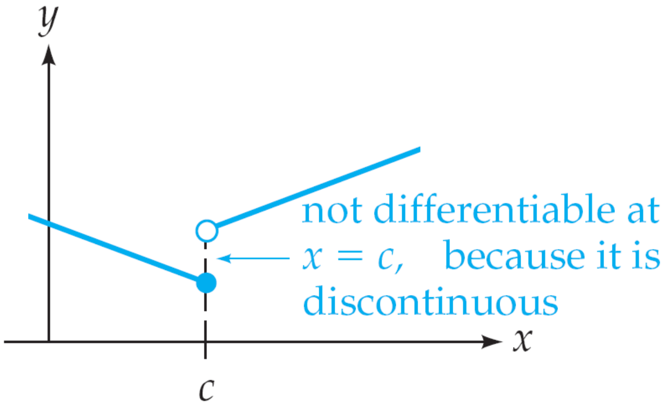
There are other reasons, besides a corner point, why a function may not be differentiable.

If a curve has a **vertical tangent** line at a point, the slope will not be defined at that *x*-value, since the slope of a vertical line is undefined.



Other Nondifferentiable Functions

If a function is differentiable, then it is continuous. Therefore, if a function is discontinuous (has a "jump") at some point, then it will not be differentiable at that *x*-value.



Other Nondifferentiable Functions

If a function f satisfies *any* of the following conditions:

1. f has a corner point at $x = c$,
2. f has a vertical tangent at $x = c$,
3. f is discontinuous at $x = c$,

then f will not be differentiable at c.