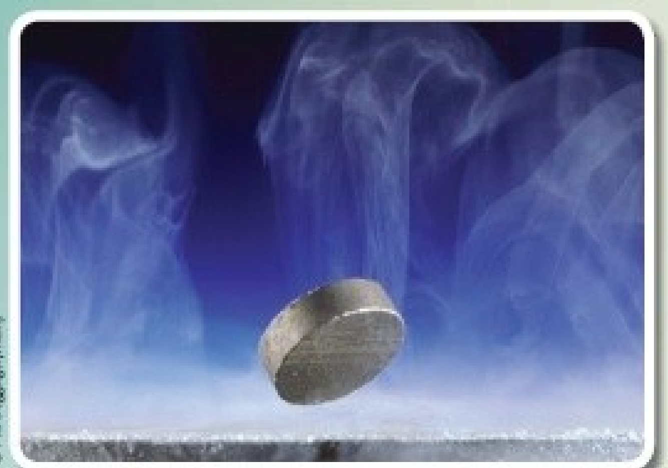
2 Derivatives and Their Uses



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2.6 THE CHAIN RULE AND THE GENERALIZED POWER RULE

Introduction

Introduction

- In this section we will learn the last of the general rules of differentiation, the **Chain Rule** for differentiating **composite functions.**

- We will then prove a very useful special case of it, the **Generalized Power Rule** for differentiating powers of functions.

Composite Functions

Composite Functions

Composite functions are simply functions of functions: the composition of f with g evaluated at x is $f(g(x))$.

Example 1 – *FINDING A COMPOSITE FUNCTION*

For $f(x) = x^2$ and $g(x) = 4 – x$, find $f(g(x))$.

Solution :

$$f(g(x)) = (4 - x)^2$$

$f(x) = x^2$ with x replaced by $g(x) = 4 – x$

The Chain Rule

The Chain Rule

If we differentiate the function $(x^2 - 5x + 1)^{10}$, we could first multiply together ten copies of $x^2 - 5x + 1$, and then differentiate the resulting polynomial. There is, however, a much easier way, using the *Chain Rule,* which shows how to differentiate a composite function of the form $f(g(x))$.

Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

To differentiate $f(g(x))$, differentiate $f(x)$, then replace each x by $g(x)$, and finally multiply by the derivative of $g(x)$

Example 3 – *DIFFERENTIATING USING THE CHAIN RULE*

Use the Chain Rule to find $\dfrac{d}{dx}(x^2 - 5x + 1)^{10}$.

Solution :

$$(x^2 - 5x + 1)^{10} \text{ is } f(g(x)) \text{ with } \begin{cases} f(x) = x^{10} & \text{Outside function} \\ g(x) = x^2 - 5x + 1 & \text{Inside function} \end{cases}$$

Since $f'(x) = 10x^9$, we have

$f'(g(x)) = 10(g(x))^9$ — $f(x) = 10x^9$ with x replaced by $g(x)$

$= 10(x^2 - 5x + 1)^9$ — Using $g(x) = x^2 - 5x + 1$

Example 3 – *Solution*

cont'd

Substituting this last expression into the Chain Rule gives:

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

Chain Rule

$$\frac{d}{dx}(x^2 - 5x + 1)^{10} = 10(x^2 - 5x + 1)^9 (2x - 5)$$

Using $f'(g(x)) = 10(x^2 - 5x + 1)^9$ and $g'(x) = 2x - 5$

Generalized Power Rule

Generalized Power Rule

Generalized Power Rule

$$\frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

To differentiate a function to a power, bring down the power as a multiplier, reduce the exponent by 1, and then multiply by the derivative of the inside function

Example 4 – *DIFFERENTIATING USING THE GENERALIZED POWER RULE*

Find $\frac{d}{dx}\sqrt{x^4 - 3x^3 - 4}.$

Solution :

$$\frac{d}{dx}(x^4 - 3x^3 - 4)^{1/2} = \frac{1}{2}(x^4 - 3x^3 - 4)^{-1/2}(4x^3 - 9x^2)$$

Inside function

Bring down the n

Power $n - 1$

Derivative of the inside function

Chain Rule in Leibniz's Notation

Chain Rule in Leibniz's Notation

A composition may be written in two parts:

$y = f(g(x))$ is equivalent to $y = f(u)$ and $u = g(x)$

The derivatives of these last two functions are:

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{du}{dx} = g'(x)$$

Chain Rule in Leibniz's Notation

The Chain Rule

$$\frac{d}{dx}\underbrace{f(g(x))}_{y} = f'(g(x)) \cdot g'(x) \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} \qquad\qquad \frac{dy}{du} \qquad \frac{du}{dx}$$

Chain Rule in Leibniz's Notation

Chain Rule in Leibniz's Notation

For $y = f(u)$ with $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In this form the Chain Rule is easy to remember, since it looks as if the *du* in the numerator and the denominator cancel:

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

However, since derivatives are not really fractions, this is only a convenient device for remembering the Chain Rule.

A Simple Example of the Chain Rule

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The derivation of the Chain Rule is rather technical, but we can show the basic idea in a simple example.

Suppose that your company produces steel, and you want to calculate your company’s total revenue in dollars per year.

You would take the revenue from a ton of steel (dollars per ton) and multiply by your company’s output (tons per year).

A Simple Example of the Chain Rule

In symbols:

$$\frac{\$}{\text{year}} = \frac{\$}{\text{ton}} \cdot \frac{\text{ton}}{\text{year}}$$

Note that "ton" cancels

If we were to express these rates as derivatives, the equation above would become the Chain Rule.