2

Derivatives and Their Uses

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2.4 THE PRODUCT AND QUOTIENT RULES

Introduction

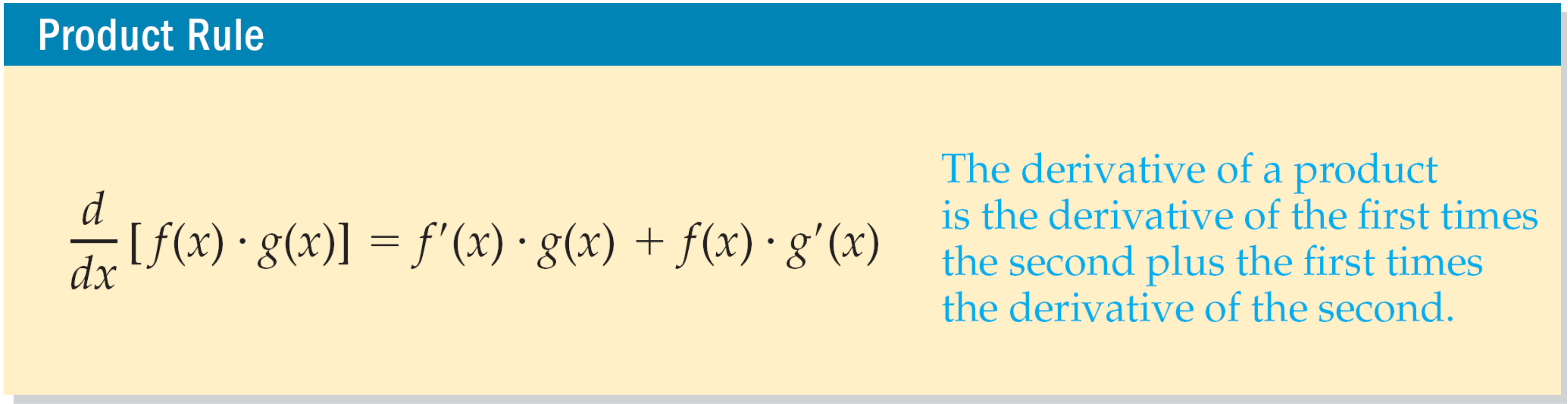
Introduction

- In this section we learn how to differentiate the *product* and *quotient* of two functions.

Product Rule

Product Rule

To differentiate the product of two functions, *f*(*x*) · *g*(*x*), we use the *Product Rule.*



Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.

Product Rule

The formula is clearer if we write the functions simply as *f* and *g*.

$$\frac{d}{dx}(f \cdot g) = \underset{\underset{\text{Derivative of the first}}{\uparrow}}{f'} \cdot \underset{\underset{\text{Second}}{\uparrow}}{g} + \underset{\underset{\text{First}}{\uparrow}}{f} \cdot \underset{\underset{\text{Derivative of the second}}{\uparrow}}{g'}$$

Example 1 – *USING THE PRODUCT RULE*

Use the Product Rule to calculate $\frac{d}{dx}(x^3 \cdot x^5)$.

Solution:

$$\frac{d}{dx}(x^3 \cdot x^5) = \underset{\substack{\uparrow \\ \text{Derivative} \\ \text{of the first}}}{3x^2} \cdot \underset{\substack{\uparrow \\ \text{Second}}}{x^5} + \underset{\substack{\uparrow \\ \text{First}}}{x^3} \cdot \underset{\substack{\uparrow \\ \text{Derivative} \\ \text{of the second}}}{5x^4}$$

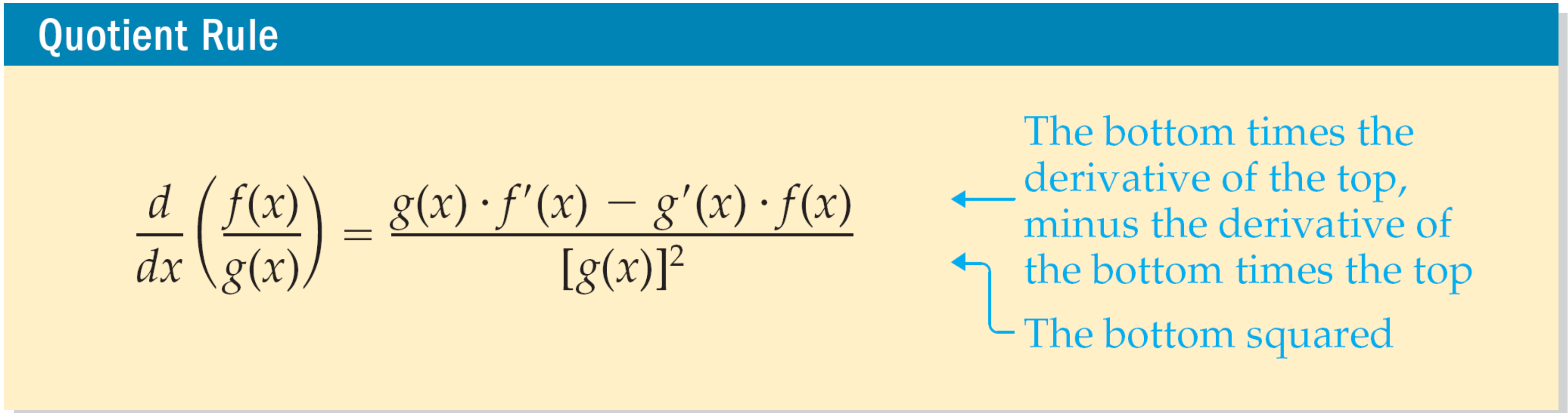
$$= 3x^7 + 5x^7$$

$$= 8x^7$$

Quotient Rule

Quotient Rule

The *Quotient Rule* shows how to differentiate a quotient of two functions.



Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot f'(x) - g'(x) \cdot f(x)}{[g(x)]^2}$$

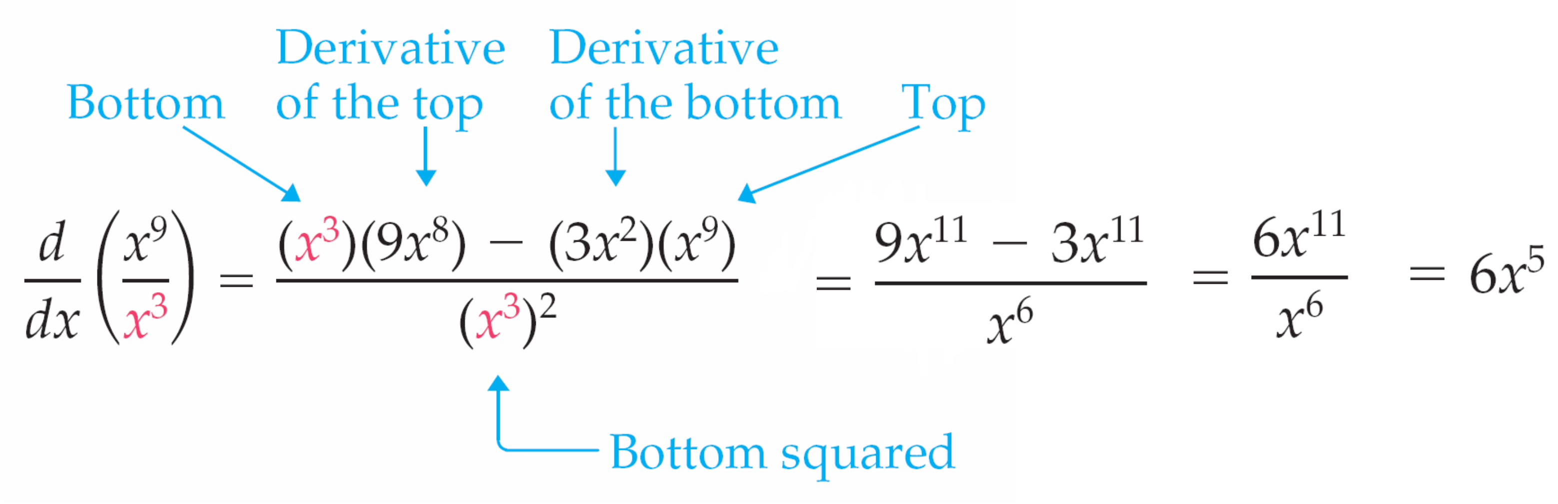
The bottom times the derivative of the top, minus the derivative of the bottom times the top

The bottom squared

Example 4 – *USING THE QUOTIENT RULE*

Use the Quotient Rule to find $\frac{d}{dx}\left(\frac{x^9}{x^3}\right)$.

Solution:



$$\frac{d}{dx}\left(\frac{x^9}{x^3}\right) = \frac{(x^3)(9x^8) - (3x^2)(x^9)}{(x^3)^2} = \frac{9x^{11} - 3x^{11}}{x^6} = \frac{6x^{11}}{x^6} = 6x^5$$

Marginal Average Cost

Marginal Average Cost

It is often useful to calculate not just the *total* cost of producing *x* units of some product, but also the **average cost per unit,** denoted *AC*(*x*), which is found by dividing the total cost *C*(*x*) by the number of units *x*.

$$AC(x) = \frac{C(x)}{x}$$

Average cost per unit is total cost divided by the number of units

Marginal Average Cost

The derivative of the average cost function is called the **marginal average cost,** *MAC.*

$$MAC(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right]$$

Marginal average cost is the derivative of average cost

Marginal Average Cost

Marginal average revenue, *MAR,* and **marginal average profit,** *MAP,* are defined similarly as the derivatives of average revenue per unit, $\frac{R(x)}{x}$, and average profit per unit, $\frac{P(x)}{x}$.

$$MAR(x) = \frac{d}{dx}\left[\frac{R(x)}{x}\right]$$

Marginal average revenue is the derivative of average revenue $\frac{R(x)}{x}$

$$MAP(x) = \frac{d}{dx}\left[\frac{P(x)}{x}\right]$$

Marginal average profit is the derivative of average profit $\frac{P(x)}{x}$

Example 8 – *FINDING AND INTERPRETING MARGINAL AVERAGE COST*

POD, or *printing on demand*, is a recent development in publishing that makes it feasible to print small quantities of books (even a single copy), thereby eliminating overstock and storage costs. For example, POD-publishing a typical 200-page book would cost $18 per copy, with fixed costs of $1500.

Therefore, the cost function is

$$C(x) = 18x + 1500 \qquad \text{Total cost of producing } x \text{ books}$$

a. Find the average cost function.

b. Find the marginal average cost function.

c. Find the marginal average cost at $x = 100$ and interpret your answer.

Source: e-booktime.com

Example 8 – *Solution*

a. The average cost function is

$$AC(x) = \underbrace{\frac{18x + 1500}{x}}_{\text{Total cost divided by number of units}} = \underbrace{18 + \frac{1500}{x}}_{\text{Simplifying}} = \underbrace{18 + 1500x^{-1}}_{\text{In power form}}$$

b. The *marginal* average cost is the derivative of average cost.

We could use the Quotient Rule on the first expression above, but it is easier to use the Power Rule on the last expression:

$$MAC(x) = \frac{d}{dx}(18 + 1500x^{-1}) = -1500x^{-2} = -\frac{1500}{x^2}$$

Example 8 – *Solution*

cont'd

c. Evaluating at $x = 100$:

$$MAC(100) = -\frac{1500}{100^2} \qquad -\frac{1500}{x^2} \text{ at } x = 100$$

$$= -\frac{1500}{10{,}000} = -0.15$$

Interpretation: When 100 books have been produced, the average cost per book is decreasing (because of the negative sign) by about *15 cents per additional book produced.*

This reflects the fact that while *total* costs rise when you produce more, the *average cost per unit* decreases, because of the economies of mass production.