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Derivatives and Their Uses

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SOME DIFFERENTIATION FORMULAS

Introduction

Introduction

- In this section we will learn several **rules of differentiation** that will simplify finding the derivatives of many useful functions. The rules are derived from the definition of the derivative.

- We will also learn another important use for differentiation: calculating **marginals** (marginal revenue, marginal cost, and marginal profit), which are used extensively in business and economics.

Derivative of a Constant

Derivative of a Constant

The first rule of differentiation shows how to differentiate a constant function.

Constant Rule

For any constant c,

$$\frac{d}{dx} c = 0$$

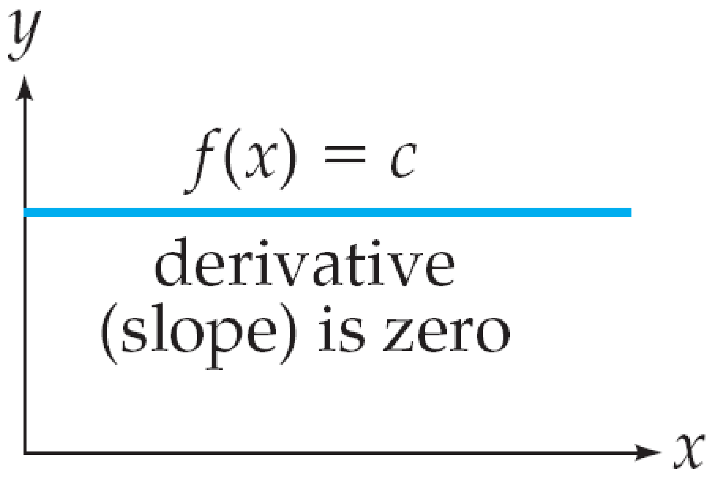
Brief Example

$$\frac{d}{dx} 7 = 0$$

In words: *the derivative of a constant is zero.*

Derivative of a Constant

This rule is obvious geometrically, as shown in the diagram: the graph of a constant function $f(x) = c$ is the horizontal line y = c. Since the slope of a horizontal line is zero, the derivative of $f(x)$ = c is zero.



Derivative of a Constant

This rule follows immediately from the definition of the derivative. The constant function *f*(*x*) = *c* has the same value *c* for *any* value of *x*, so, in particular, *f*(*x* + *h*) = *c* and *f*(*x*) = *c*. Substituting these into the definition of the derivative gives

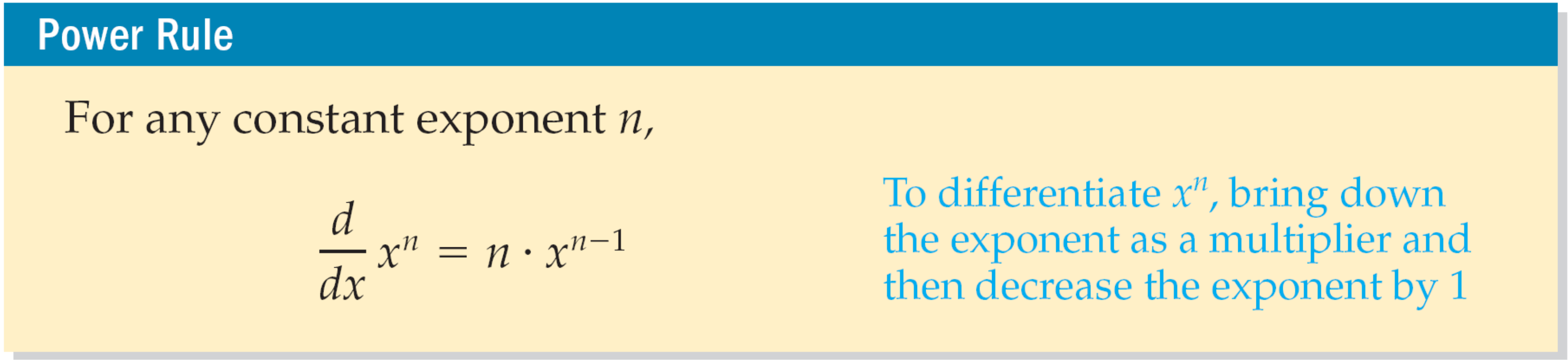
$$f'(x) = \lim_{h \to 0} \frac{\overbrace{f(x+h)}^{c} - \overbrace{f(x)}^{c}}{h} = \lim_{h \to 0} \frac{\overbrace{c - c}^{0}}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Therefore, the derivative of a constant function *f*(*x*) = *c* is *f*′(*x*) = 0.

Power Rule

Power Rule

One of the most useful differentiation formulas in all of calculus is the **Power Rule.** It tells how to differentiate powers such as x^7 or x^{100}.



Power Rule

For any constant exponent n,

$$\frac{d}{dx}x^n = n \cdot x^{n-1}$$

To differentiate x^n, bring down the exponent as a multiplier and then decrease the exponent by 1

Example 1 – *USING THE POWER RULE*

a. $$\frac{d}{dx}x^{7} = 7x^{7-1} = 7x^{6}$$

Bring down the exponent

Decrease the exponent by 1

b. $$\frac{d}{dx}x^{100} = 100x^{100-1} = 100x^{99}$$

c. $$\frac{d}{dx}x^{-2} = -2x^{-2-1} = -2x^{-3}$$

The Power Rule holds for negative exponents

d. $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

and for fractional exponents

e. $$\frac{d}{dx}x = \frac{d}{dx}x^{1} = 1x^{1-1} = x^{0} = 1$$

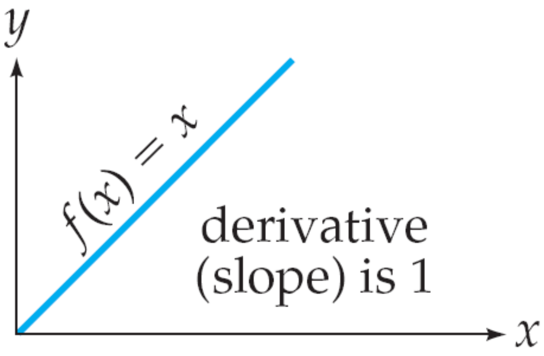
Power Rule

The derivative of *x* is used so frequently that it should be remembered separately.

$$\frac{d}{dx}x = 1$$

The derivative of x is 1

This result is obvious geometrically, as shown in the diagram.



Constant Multiple Rule

Constant Multiple Rule

Constant Multiple Rule

For any constant c,

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

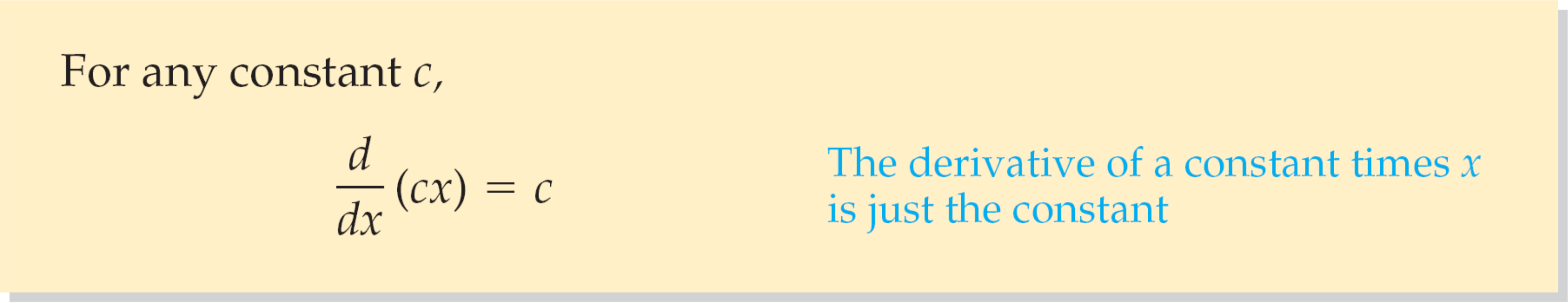
The derivative of a constant times a function is the constant times the derivative of the function

Example 2 – *USING THE CONSTANT MULTIPLE RULE*

a. $$\frac{d}{dx}5x^3 = \underset{\substack{\uparrow \\ \text{Carry along} \\ \text{the constant}}}{5} \cdot \underbrace{3x^2}_{\substack{\uparrow \\ \text{Derivative} \\ \text{of } x^3}} = 15x^2$$

b. $$\frac{d}{dx}3x^{-4} = 3(-4)x^{-5} = -12x^{-5}$$

Constant Multiple Rule



For any constant c,

$$\frac{d}{dx}(cx) = c$$

The derivative of a constant times x is just the constant

Sum Rule

Sum Rule

The *Sum Rule* extends differentiation to sums of functions. Briefly, to differentiate a *sum* of two functions, just differentiate the functions separately and add the results.

Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

Brief Example

$$\frac{d}{dx}(x^3 + x^5) = 3x^2 + 5x^4$$

Sum Rule

A similar rule holds for the *difference* of two functions,

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

The derivative of a difference is the difference of the derivatives

These two rules may be combined:

Sum-Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Use both upper signs
or both lower signs

Example 5 – *USING THE SUM-DIFFERENCE RULE*

a. $\frac{d}{dx}(x^3 - x^5) = 3x^2 - 5x^4$

Derivatives taken separately

b. $\frac{d}{dx}(5x^{-2} - 6x^{1/3} + 4) = -10x^{-3} - 2x^{-2/3}$

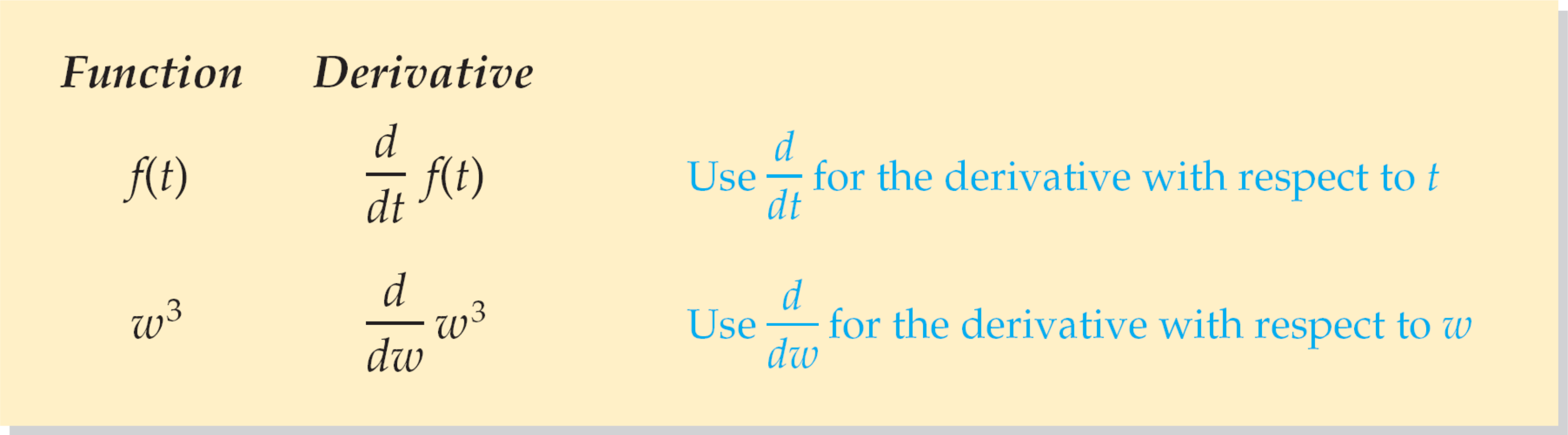
The constant 4 has derivative 0

Leibniz’s Notation and Evaluation of Derivatives

Leibniz’s Notation and Evaluation of Derivatives

- Leibniz’s derivative notation $\frac{d}{dx}$ is often read *the derivative with respect to x* to emphasize that the independent variable is *x*.

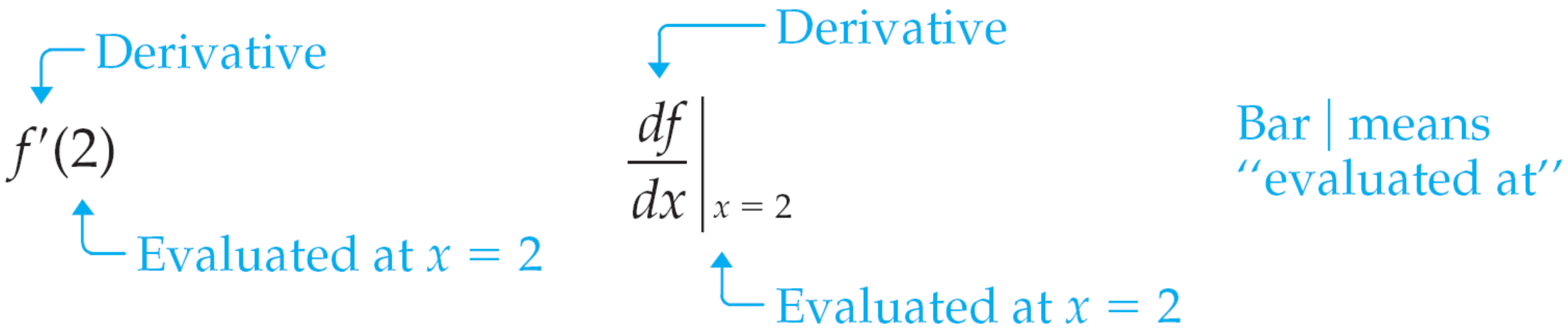
- To differentiate a function of some *other* variable, the *x* in $\frac{d}{dx}$ replaced by the other variable. For example:



Function	Derivative	
$f(t)$	$\frac{d}{dt} f(t)$	Use $\frac{d}{dt}$ for the derivative with respect to t
w^3	$\frac{d}{dw} w^3$	Use $\frac{d}{dw}$ for the derivative with respect to w

Leibniz’s Notation and Evaluation of Derivatives

The following two notations both mean the derivative *evaluated* at $x = 2$.



Derivative

$f'(2)$

Evaluated at $x = 2$

Derivative

$$\left.\frac{df}{dx}\right|_{x=2}$$

Evaluated at $x = 2$

Bar | means “evaluated at”

Example 6 – *EVALUATING A DERIVATIVE*

If $f(x) = 2x^3 - 5x^2 + 7$, find $f'(2)$.

Solution:

$f'(x) = 6x^2 - 10x$

First differentiate

$f'(2) = 6 \cdot 2^2 - 10 \cdot 2 = 24 - 20 = 4$

Then evaluate

Derivatives in Business and Economics: Marginals

Derivatives in Business and Economics: Marginals

There is another interpretation for the derivative, one that is particularly important in business and economics.

Suppose that a company has calculated its revenue, cost, and profit functions, as defined below.

$$R(x) = \left(\begin{array}{c}\text{Total revenue (income)}\\ \text{from selling } x \text{ units}\end{array}\right) \qquad \text{Revenue function}$$

$$C(x) = \left(\begin{array}{c}\text{Total cost of}\\ \text{producing } x \text{ units}\end{array}\right) \qquad \text{Cost function}$$

$$P(x) = \left(\begin{array}{c}\text{Total profit from producing}\\ \text{and selling } x \text{ units}\end{array}\right) \qquad \text{Profit function}$$

Derivatives in Business and Economics: Marginals

The term **marginal cost** means the additional cost of producing one more unit, $C(x + 1) - C(x)$, which may be written $\frac{C(x+1) - C(x)}{1}$, which is just the difference quotient $\frac{C(x+h) - C(x)}{h}$ with $h = 1$.

If many units are being produced, then $h = 1$ is a relatively small number compared with x, so this difference quotient may be approximated by its limit as $h \to 0$, that is, by the *derivative* of the cost function.

Derivatives in Business and Economics: Marginals

In view of this approximation, in calculus the marginal cost is *defined* to be the derivative of the cost function:

$$MC(x) = C'(x)$$ Marginal cost is the derivative of cost

The **marginal revenue** function *MR*(*x*) and the **marginal profit** function *MP*(*x*) are similarly defined as the derivatives of the revenue and cost functions.

$$MR(x) = R'(x)$$ Marginal revenue is the derivative of revenue

$$MP(x) = P'(x)$$ Marginal profit is the derivative of profit

Derivatives in Business and Economics: Marginals

All of this can be summarized very briefly: “marginal” means “derivative of.” We now have three interpretations for the derivative: *slopes, instantaneous rates of change,* and *marginals.*

Example 8 – *FINDING AND INTERPRETING MARGINAL COST*

The Pocket EZCie is a miniature key chain flashlight based on LED (light-emitting diode) technology. The cost function (the total cost of producing *x* Pocket EZCies) is

$$C(x) = 8\sqrt[4]{x^3} + 300$$

dollars, where *x* is the number of Pocket EZCies produced.

a. Find the marginal cost function $MC(x)$.

b. Find the marginal cost when 81 Pocket EZCies have been produced and interpret your answer.

Example 8 – *Solution*

a. The marginal cost function is the derivative of the cost function $C(x) = 8x^{3/4} + 300,$ so

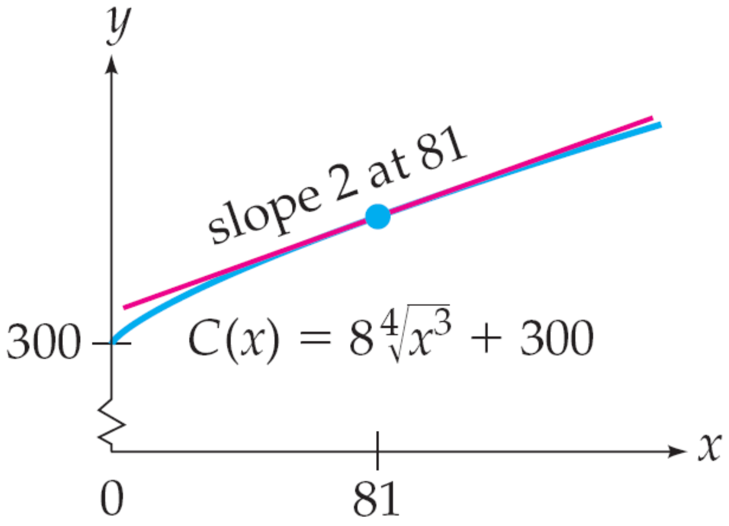
$$MC(x) = 6x^{-1/4} = \frac{6}{\sqrt[4]{x}} \qquad \text{Derivative of } C(x)$$

b. To find the marginal cost when 81 Pocket EZCies have been produced, we evaluate the marginal cost function *MC*(*x*) at *x* = 81:

$$MC(81) = \frac{6}{\sqrt[4]{81}} = \frac{6}{3} = 2 \qquad MC(x) = \frac{6}{\sqrt[4]{x}} \text{ evaluated at } x = 81$$

Example 8 – *Solution*

cont'd



Cost function showing the marginal cost (slope) at *x* = 81

Interpretation: When 81 Pocket EZCies have been produced, the marginal cost is $2, meaning that to produce one more Pocket EZCie costs about $2.

Source: EZCie Manufacturing

Functions as Single Objects

Functions as Single Objects

You have looked at functions and graphs as collections of individual points, to be plotted one at a time. Now, however, we are operating on *whole functions* all at once.

In calculus, the basic objects of interest are *functions,* and a function should be thought of as a *single* object.

Derivatives on a Graphing Calculator

Derivatives on a Graphing Calculator

- Graphing calculators have an operation called NDERIV, standing for **numerical derivative,** which gives an *approximation* of the derivative of a function.

- Most do so by evaluating the **symmetric difference quotient,** $\dfrac{f(x+h)-f(x-h)}{2h}$ for a small value of *h*, such as *h* = 0.001.

- The numerator represents the change in the function when *x* changes by 2*h* (from *x* – *h* to *x* + *h*), and the denominator divides by this change in *x*.

Derivatives on a Graphing Calculator

- Geometrically, the symmetric difference quotient gives the slope of the secant line through two points on the curve *h* units on either side of the point at *x*.

- While NDERIV usually approximates the derivative quite closely, it sometimes gives erroneous results. For this reason, using a graphing calculator effectively requires an understanding of both the calculus that underlies it and the technology that limits it.