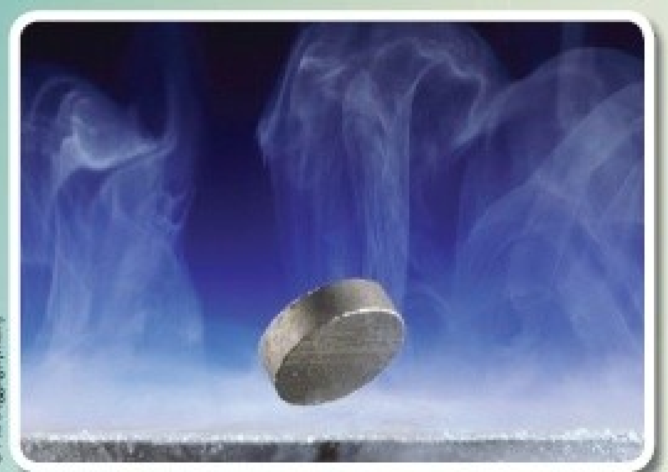
2 Derivatives and Their Uses



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2.1 LIMITS AND CONTINUITY

Introduction

Introduction

- This section introduces the **derivative,** one of the most important concepts in all of calculus.

- We begin by discussing two preliminary topics, **limits** and **continuity,** both of which will be treated intuitively rather than formally.

Limits

Limits

The word “limit” is used in everyday conversation to describe the ultimate behavior of something, as in the “limit of one’s endurance” or the “limit of one’s patience.”

In mathematics, the word “limit” has a similar but more precise meaning.

The notation $x \to 3$ (read: “*x* approaches 3”) means that *x* takes values *arbitrarily close to 3 without ever equaling 3*.

Limits

Given a function *f*(*x*), if *x* approaching 3 causes the function to take values approaching (or equalling) some particular number, such as 10, then we will call 10 *the limit of the function* and write

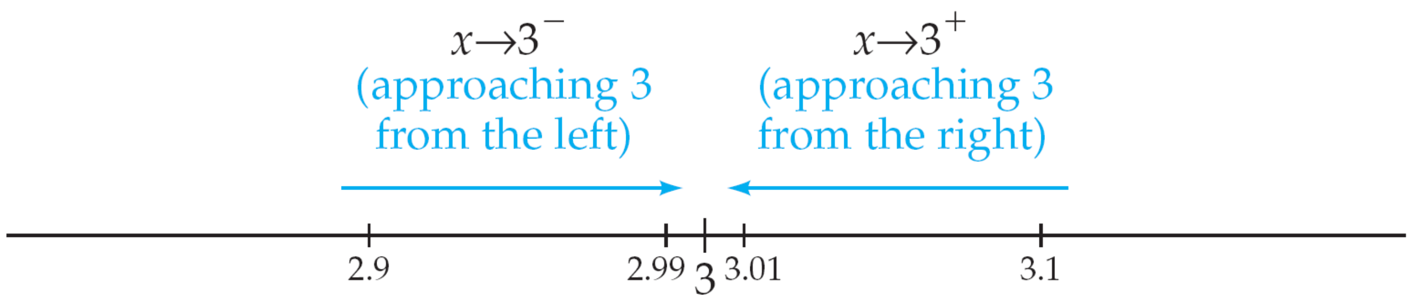
$$\lim_{x \to 3} f(x) = 10$$

Limit of $f(x)$ as x approaches 3 is 10

In practice, the two simplest ways we can approach 3 are *from the left* or *from the right.*

Limits

For example, the numbers 2.9, 2.99, 2.999, ... approach 3 *from the left*, which we denote by $x \to 3^-$, and the numbers 3.1, 3.01, 3.001, ... approach 3 *from the right*, denoted by $x \to 3^+$. Such limits are called *one-sided limits.*



Example 1 – *FINDING A LIMIT BY TABLES*

Use tables to find $\lim_{x \to 3} (2x + 4)$. Limit of $2x + 4$ as x approaches 3

Solution :

We make two tables, as shown below, one with *x* approaching 3 *from the left*, and the other with *x* approaching 3 *from the right.*

Approaching 3 *from the left*

x	$2x + 4$
2.9	9.8
2.99	9.98
2.999	9.998

Limit is 10

This table shows $\lim_{x \to 3^-} (2x + 4) = 10$

Approaching 3 *from the right*

x	$2x + 4$
3.1	10.2
3.01	10.02
3.001	10.002

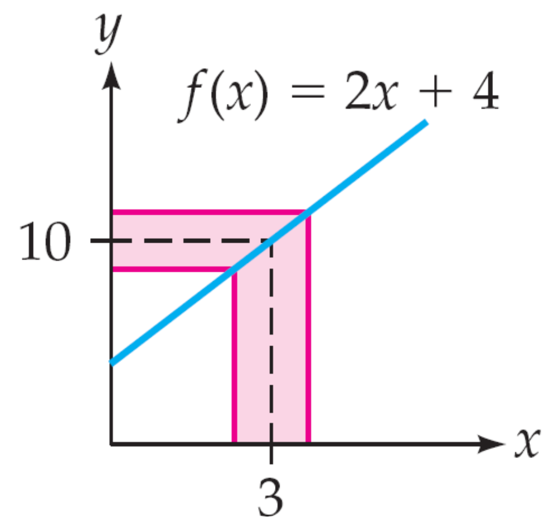
Limit is 10

This table shows $\lim_{x \to 3^+} (2x + 4) = 10$

Example 1 – *Solution*

cont'd

Choosing *x*-values even closer to 3 (such as 2.9999 or 3.0001) would result in values of 2*x* + 4 *even closer to 10,* so that both one-sided limits equal 10:

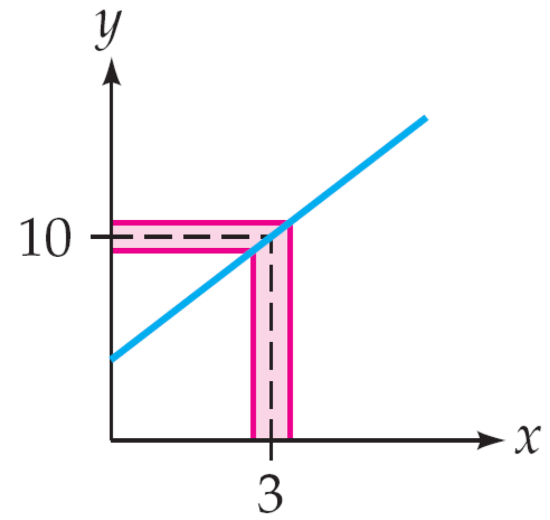


$$\lim_{x \to 3^-} (2x + 4) = 10 \qquad \text{and} \qquad \lim_{x \to 3^+} (2x + 4) = 10$$

Example 1 – *Solution*

cont'd

Since approaching 3 from *either* side causes 2*x* + 4 to approach the same number, 10, we may state that *the limit* is 10:



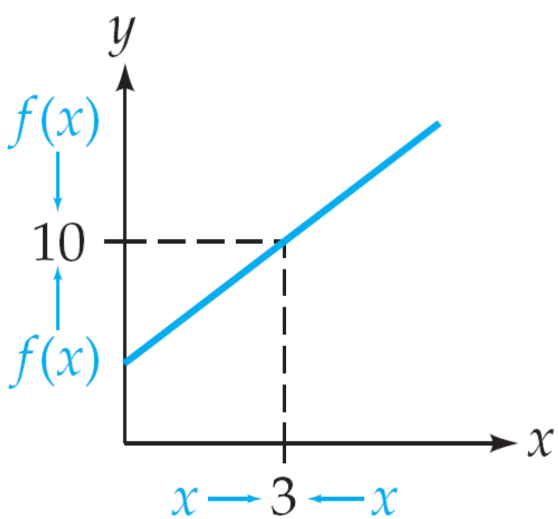
$$\lim_{x \to 3} (2x + 4) = 10$$ Limit of $2x + 4$ as x approaches 3 is 10

Example 1 – *Solution*

cont'd

This limit says that as x approaches 3 from *either side*, or even *alternating sides*, as in 2.9, 3.01, 2.999, 3.0001, ... , the values $2x + 4$ of will approach 10.

This may be seen in the succession of graphs shown below: as $x \to 3$ (on the x-axis), $f(x) \to 10$ (on the y-axis).



Limits

Limits

The statement

$$\lim_{x \to c} f(x) = L \qquad \text{Limit of } f(x) \text{ as } x \text{ approaches } c \text{ is } L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking values of x sufficiently close (but not equal) to c.

The one-sided limits

$$\lim_{x \to c^-} f(x) = L \qquad \text{Limit of } f(x) \text{ as } x \text{ approaches } c \text{ from the } \textit{left} \text{ is } L$$

$$\lim_{x \to c^+} f(x) = L \qquad \text{Limit of } f(x) \text{ as } x \text{ approaches } c \text{ from the } \textit{right} \text{ is } L$$

have similar meanings but with the x-values restricted to, respectively, $x < c$ and $x > c$.

Limits

The limit $\lim_{x \to c} f(x)$ is sometimes called a *two-sided limit* to distinguish it from one-sided limits. A limit may fail to exist, but if the limit *does* exist, it must be a *single number*.

If both one-sided limits exist and have the same value, then the (two-sided) limit will exist and *have this same value*.

The correct limit in $\lim_{x \to 3} (2x + 4)$ could have been found simply by *evaluating* the function at $x = 3$:

$$f(3) = 2 \cdot 3 + 4 = 10$$

$f(x) = 2x + 4$ evaluated at $x = 3$ gives the correct limit, 10

Limits

Which limits can be evaluated by **direct substitution** (for example $\lim_{x\to 3}(2x+y)$) and which cannot (for example $\lim_{x\to 0}(1+x)^{1/x}$) ?

The answer comes from the following "Rules of Limits."

Rules of Limits

For any constants a and c, and any positive integer n:

1. $\lim_{x\to c} a = a$	The limit of a constant is just the constant
2. $\lim_{x\to c} x^n = c^n$	The limit of a power is the power of the limit
3. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ $\quad$ ($c > 0$ if n is even)	The limit of a root is the root of the limit
4. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then	
a. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$	The limit of a sum is the sum of the limits
b. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$	The limit of a difference is the difference of the limits
c. $\lim_{x\to c}[f(x)\cdot g(x)] = [\lim_{x\to c} f(x)]\cdot[\lim_{x\to c} g(x)]$	The limit of a product is the product of the limits
d. $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ $\quad$ (if $\lim_{x\to c} g(x) \neq 0$)	The limit of a quotient is the quotient of the limits

Limits

These rules, which may be proved from the definition of limit, can be summarized as follows.

Summary of Rules of Limits

For functions composed of additions, subtractions, multiplications, divisions, powers, and roots, limits may be evaluated by direct substitution, provided that the resulting expression is defined.

$$\lim_{x \to c} f(x) = f(c)$$ Limit evaluated by direct substitution

The *rules of limits* and also the above *summary* hold for one-sided limits as well as for regular (two-sided) limits.

Example 3 – *FINDING LIMITS BY DIRECT SUBSTITUTION*

a. $\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$

Direct substitution of $x = 4$
using Rule 3 or the Summary

b. $\lim_{x \to 6} \frac{x^2}{x+3} = \frac{6^2}{6+3} = \frac{36}{9} = 4$

Direct substitution of $x = 6$
(Rules 4, 2, and 1 or the Summary)

Limits Involving Infinity

Limits Involving Infinity

- A limit statement such as $\lim_{x \to c} f(x) = \infty$ (the symbol is read "infinity") does *not* mean that the function takes values near "infinity," since there is no number "infinity."
  It means, instead, that the values of the function *become arbitrarily large* (the graph rises arbitrarily high) near the number *c*.

- Similarly, the limit statement $\lim_{x \to c} f(x) = -\infty$ means that the values of the function *become arbitrarily small* (the graph falls arbitrarily low) near the number *c*.

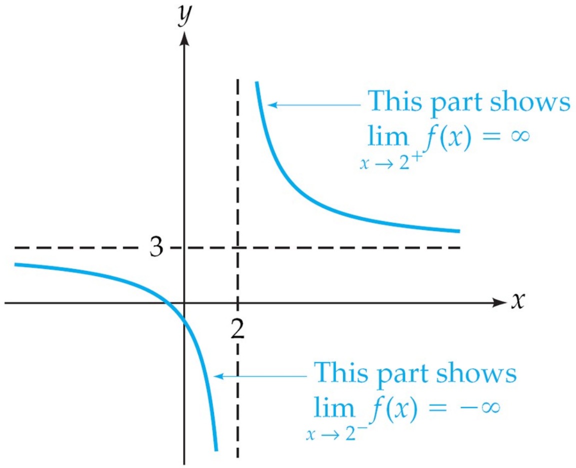
  Such statements, which may also be written with one-sided limits, mean that the graph has a *vertical asymptote* at $x = c$.

Example 6 – *FINDING LIMITS INVOLVING* ±∞

Describe the asymptotic behavior of $f(x) = \frac{3x + 2}{x - 2}$ using limits involving $\pm\infty$.

Solution :

The graph of $f(x) = \frac{3x + 2}{x - 2}$ is shown below.



Graph of $f(x) = \frac{3x + 2}{x - 2}$

Example 6 – *Solution*

cont'd

It is undefined at $x = 2$ (the denominator would be zero), as indicated by the vertical dashed line. Notice that as x approaches 2 *from the right,* the curve rises arbitrarily *high,* and as x approaches 2 *from the left,* the curve falls arbitrarily *low,* as expressed in the limit statements below.

The tables below show these results numerically.

x	$\frac{3x+2}{x-2}$
2.1	83
2.01	803
2.001	8003

This table shows $\lim_{x \to 2^+} f(x) = \infty$

x	$\frac{3x+2}{x-2}$
1.9	−77
1.99	−797
1.999	−7997

This table shows $\lim_{x \to 2^-} f(x) = -\infty$

Example 6 – *Solution*

cont'd

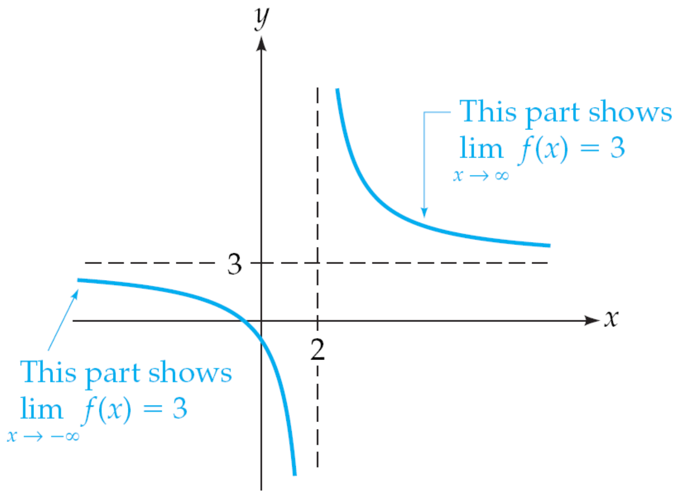
Since the limit from one side was ∞ and from the other was $-\infty$, the (two sided) limit $\lim_{x \to 2} f(x)$ *does not exist.* (If *both* one-sided limits had yielded $-\infty$, we could have stated $\lim_{x \to 2} f(x) = \infty$, and if both had yielded $-\infty$ we could have stated $\lim_{x \to 2} f(x) = -\infty$.)

For *x*-values *arbitrarily far out to the right* (denoted $x \to \infty$), the curve levels off approaching height 3, which we express as $\lim_{x \to \infty} f(x) = 3$, *x*-values *arbitrarily far out to the left* (denoted $x \to -\infty$), the curve again levels off at height 3, which we express as $\lim_{x \to -\infty} f(x) = 3.$

Example 6 – *Solution*

cont'd

These results are shown below both graphically and numerically: the graph has a *horizontal asymptote* at $y = 3$ and the tables have (rounded) function values approaching 3.



Graph of $f(x) = \dfrac{3x+2}{x-2}$

x	$\dfrac{3x+2}{x-2}$
100	3.082
1000	3.008
10000	3.001

This table shows $\lim_{x \to \infty} f(x) = 3$

x	$\dfrac{3x+2}{x-2}$
−100	2.922
−1000	2.992
−10000	2.999

This table shows $\lim_{x \to -\infty} f(x) = 3$

Limits Involving Infinity

Limits Involving Infinity

$\lim_{x \to c^+} f(x) = \infty$	means that the values of $f(x)$ grow arbitrarily large as x approaches c from the right
$\lim_{x \to c^-} f(x) = \infty$	means that the values of $f(x)$ grow arbitrarily large as x approaches c from the left
$\lim_{x \to c} f(x) = \infty$	means that *both* of the above statements are true

Similar statements hold if ∞ is replaced by $-\infty$ and the words "arbitrarily large" by "arbitrarily small."

$\lim_{x \to \infty} f(x) = L$	means that the values of $f(x)$ become arbitrarily close to the number L as x becomes arbitrarily large
$\lim_{x \to -\infty} f(x) = L$	means that the values of $f(x)$ become arbitrarily close to the number L as x becomes arbitrarily small

Limits of Functions of Two Variables

Limits of Functions of Two Variables

Some limits involve two variables, with only one variable approaching a limit.

Example 7 – *FINDING A LIMIT OF A FUNCTION OF TWO VARIABLES*

Find $\lim_{h \to 0} (x^2 + xh + h^2)$.

Solution:

Only *h* is approaching zero, so *x* remains unchanged. Since the function involves only powers of *h,* we may evaluate the limit by direct substitution of *h* = 0:

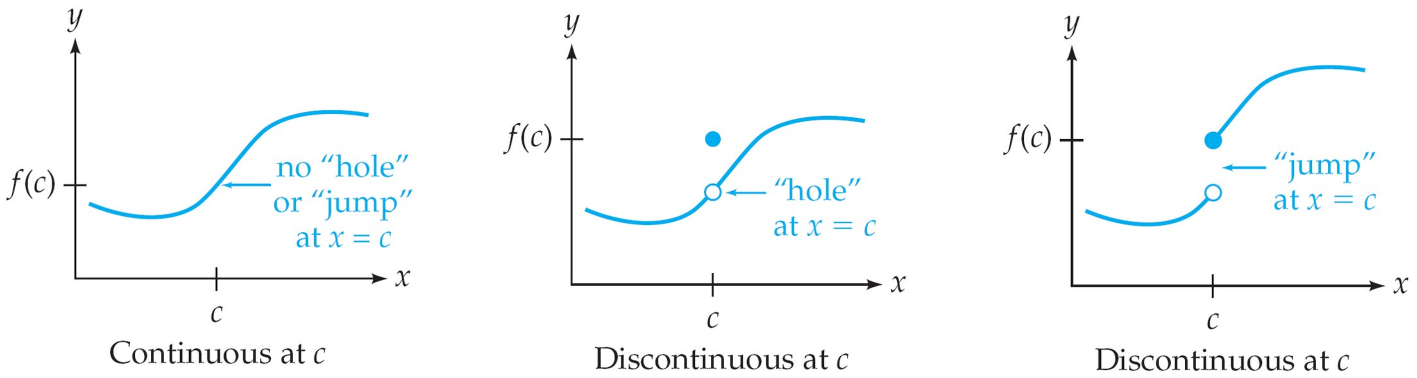
$$\lim_{h \to 0} (x^2 + xh + h^2) = x^2 + \underbrace{x \cdot 0}_{0} + \underbrace{0^2}_{0} = x^2$$

Continuity

Continuity

Intuitively, a function is said to be **continuous** at *c* if its graph passes through the point at *x* = *c without a "hole" or a "jump*."

For example, the first function below is *continuous* at *c* (it has no hole or jump at *x* = *c*) while the second and third are **discontinuous** at *c*.



Continuity

In other words, a function is *continuous at c* if the curve *approaches the point at* $x = c$, which may be stated in terms of limits:

$$\lim_{x \to c} f(x) = f(c)$$

Height of the *curve* approaches the height of the *point*

This equation means that the quantities on both sides must exist and be *equal*.

Continuity

Continuity

A function f is continuous at c if the following three conditions hold:

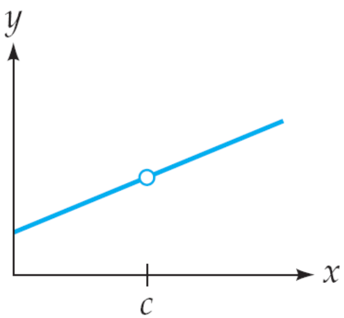
1. $f(c)$ is defined — Function is *defined* at c
2. $\lim_{x \to c} f(x)$ exists — Left and right limits exist and agree
3. $\lim_{x \to c} f(x) = f(c)$ — Limit and value *at c* agree

f is *discontinuous* at c if one or more of these conditions *fails* to hold.

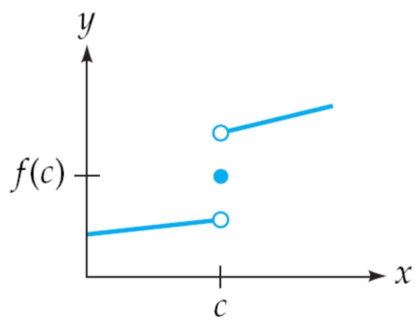
Condition 3, which is just the statement that the expressions in Conditions 1 and 2 are equal to each other, may by itself be taken as the definition of continuity.

Example 8 – *FINDING DISCONTINUITIES FROM A GRAPH*

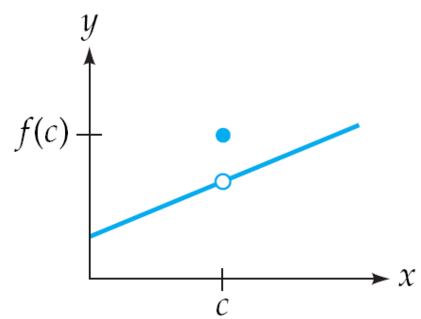
Each function below is *discontinuous at c* for the indicated reason.



$f(c)$ is not defined
(the point at $x = c$ is missing)



$\lim_{x \to c} f(x)$ does not exist
(the left and right limits do not agree)



$\lim_{x \to c} f(x) \neq f(c)$
(the limit exists and $f(c)$ is defined, but they are *not* equal)

Continuity on Intervals

Continuity on Intervals

- A function is continuous on an *open interval* (a, b) if it is continuous at each point of the interval.

- A function is continuous on a *closed interval* [a, b] if it is continuous on the open interval (a, b) and has "one-sided continuity" at the endpoints:

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

- A function that is continuous on the entire real line $(-\infty, \infty)$ is said to be *continuous everywhere,* or simply *continuous*.

Which Functions Are Continuous?

Which Functions Are Continuous?

- *Linear* and *quadratic* functions are continuous, since their graphs are, respectively, straight lines and parabolas, with no holes or jumps.

- *Exponential* functions are continuous and *logarithmic* functions are continuous on their domains.

Which Functions Are Continuous?

Continuous Functions

If functions f and g are continuous at c, then the following are also continuous at c:

1. $f \pm g$ — Sums and differences of continuous functions are continuous
2. $a \cdot f$ [for any constant a] — Constant multiples of continuous functions are continuous
3. $f \cdot g$ — Products of continuous functions are continuous
4. f/g [if $g(c) \neq 0$] — Quotients of continuous functions are continuous
5. $f(g(x))$ [for f continuous at $g(c)$] — Compositions of continuous functions are continuous

Every polynomial function is continuous.

Every rational function is continuous except where the denominator is zero.

Example 9 – *DETERMINING CONTINUITY*

Determine whether each function is continuous or discontinuous. If discontinuous, state *where* it is discontinuous.

a. $f(x) = x^3 - 3x^2 - x + 3$

b. $f(x) = \dfrac{1}{(x + 1)^2}$

c. $f(x) = e^{x^2-1}$

Example 9 – *Solution*

cont'd

a. The first function is continuous since it is a polynomial.

b. The second function is discontinuous at $x = -1$

(where the denominator is zero). It is continuous at all other x-values.

c. The third function is continuous since it is the composition of the exponential function e^x and the polynomial $x^2 - 1$.

Relationship Between Limits and Continuity

Relationship Between Limits and Continuity

A function is continuous at *c* if its graph passes through the point at *x* = *c* without a "hole" or a "jump."

We then used limits to define continuity more precisely, saying that to be continuous at *c*, *f* must satisfy

$$\lim_{x \to c} f(x) = f(c)$$

This is just the equation that we can find which limits may be evaluated by *direct substitution*.

Relationship Between Limits and Continuity

We may now use continuity to restate this result more succinctly:

The limit $\lim_{x \to c} f(x)$ may be evaluated by direct substitution if and only if *f* is continuous at *c*.