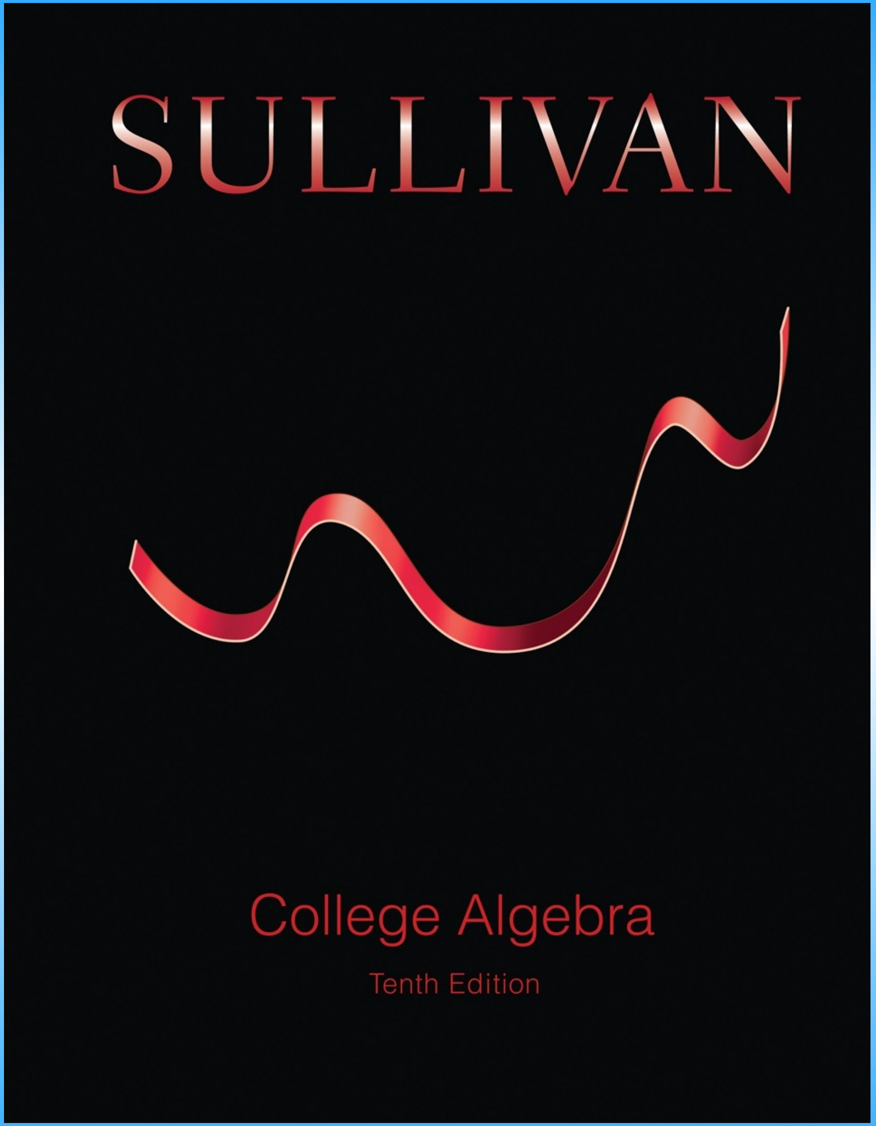
Chapter R

Section 4



R.4 Polynomials

OBJECTIVES

Recognize Monomials

Definition

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial.

Recognize Polynomials

Definition

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, $a_n x^n$ is called the **leading term**, and n is the **degree** of the polynomial.

Example

Examples of Polynomials

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

Add and Subtract Polynomials

Example

Adding Polynomials

Find the sum of the polynomials:

$$8x^3 - 2x^2 + 6x - 2 \quad \text{and} \quad 3x^4 - 2x^3 + x^2 + x$$

Solution

We shall find the sum in two ways.

Horizontal Addition: The idea here is to group the like terms and then combine them.

$$\begin{aligned}&(8x^3 - 2x^2 + 6x - 2) + (3x^4 - 2x^3 + x^2 + x)\\&= 3x^4 + (8x^3 - 2x^3) + (-2x^2 + x^2) + (6x + x) - 2\\&= 3x^4 + 6x^3 - x^2 + 7x - 2\end{aligned}$$

Vertical Addition: The idea here is to vertically line up the like terms in each polynomial and then add the coefficients.

$$\begin{array}{rrrrr} x^4 & x^3 & x^2 & x^1 & x^0 \\ & 8x^3 & -\,2x^2 & +\,6x & -\,2 \\ +\,3x^4 & -\,2x^3 & +\,x^2 & +\,x & \\ \hline 3x^4 & +\,6x^3 & -\,x^2 & +\,7x & -\,2 \end{array}$$

Example

Subtracting Polynomials

Find the difference: $(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$

Solution

Horizontal Subtraction:

$$(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$$

$$= 3x^4 - 4x^3 + 6x^2 - 1 + \underbrace{(-2x^4 + 8x^2 + 6x - 5)}$$

Be sure to change the sign of each term in the second polynomial.

$$= (3x^4 - 2x^4) + (-4x^3) + (6x^2 + 8x^2) + 6x + (-1 - 5)$$

↑ Group like terms.

$$= x^4 - 4x^3 + 14x^2 + 6x - 6$$

Solution continued

Vertical Subtraction: We line up like terms, change the sign of each coefficient of the second polynomial, and add.

$$
\begin{array}{rrrrrrcrrrrrr}
 & x^4 & x^3 & x^2 & x^1 & x^0 & & & x^4 & x^3 & x^2 & x^1 & x^0 \\
 & 3x^4 & -\,4x^3 & +\,6x^2 & & -\,1 & = & & 3x^4 & -\,4x^3 & +\,6x^2 & & -\,1 \\
-\,[& 2x^4 & & -\,8x^2 & -\,6x & +\,5\,] & = & + & -2x^4 & & +\,8x^2 & +\,6x & -\,5 \\
\hline
 & & & & & & & & x^4 & -\,4x^3 & +\,14x^2 & +\,6x & -\,6
\end{array}
$$

Multiply Polynomials

Example

Multiplying Polynomials

Find the product: $(2x + 5)(x^2 - x + 2)$

Solution

Horizontal Multiplication:

$$(2x + 5)(x^2 - x + 2) = 2x(x^2 - x + 2) + 5(x^2 - x + 2)$$

↑ Distributive Property

$$= (2x \cdot x^2 - 2x \cdot x + 2x \cdot 2) + (5 \cdot x^2 - 5 \cdot x + 5 \cdot 2)$$

↑ Distributive Property

$$= (2x^3 - 2x^2 + 4x) + (5x^2 - 5x + 10)$$

↑ Law of Exponents

$$= 2x^3 + 3x^2 - x + 10$$

↑ Combine like terms.

Solution continued

Vertical Multiplication: The idea here is very much like multiplying a two-digit number by a three-digit number.

$$
\begin{array}{rrrrl}
 & x^2 & - x & + 2 & \\
 & & 2x & + 5 & \\
\hline
2x^3 & - 2x^2 & + 4x & & \text{This line is } 2x(x^2 - x + 2). \\
(+) & 5x^2 & - 5x & + 10 & \text{This line is } 5(x^2 - x + 2). \\
\hline
2x^3 & + 3x^2 & - x & + 10 & \text{Sum of the above two lines}
\end{array}
$$

Know Formulas for Special Products

Example

Using FOIL

(a) $(x - 3)(x + 3) = x^2 + 3x - 3x - 9 = x^2 - 9$

F O I L

(b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x + 3)(x + 1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$

$$(x - a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

Example

Using Special Product Formulas

(a) $(x-5)(x+5) = x^2 - 5^2 = x^2 - 25$ — Difference of two squares

(b) $(x+7)^2 = x^2 + 2\cdot 7\cdot x + 7^2 = x^2 + 14x + 49$ — Square of a binomial

(c) $(2x+1)^2 = (2x)^2 + 2\cdot 1\cdot 2x + 1^2 = 4x^2 + 4x + 1$ — Notice that we used $2x$ in place of x in formula (3a).

(d) $(3x-4)^2 = (3x)^2 - 2\cdot 4\cdot 3x + 4^2 = 9x^2 - 24x + 16$ — Replace x by $3x$ in formula (3b).

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \qquad \textbf{(4a)}$$

$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \qquad \textbf{(4b)}$$

Example

Forming the Difference of Two Cubes

$$\begin{aligned}(x-1)(x^2+x+1) &= x(x^2+x+1) - 1(x^2+x+1) \\ &= x^3+x^2+x-x^2-x-1 \\ &= x^3-1\end{aligned}$$

Example

Forming the Sum of Two Cubes

$$\begin{aligned}(x+2)(x^2-2x+4) &= x(x^2-2x+4)+2(x^2-2x+4)\\ &= x^3-2x^2+4x+2x^2-4x+8\\ &= x^3+8\end{aligned}$$

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \qquad \textbf{(5)}$$

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \qquad \textbf{(6)}$$

Divide Polynomials Using Long Division

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

Example

Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Solution

Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

Step 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{r l} & 3x \\ x^2 + 1 \big) & \overline{3x^3 + 4x^2 + x + 7} \end{array}$$

Step 2: Multiply $3x$ by $x^2 + 1$, and enter the result below the dividend.

$$\begin{array}{r l} & 3x \\ x^2 + 1 \big) & \overline{3x^3 + 4x^2 + x + 7} \\ & \underline{3x^3 + 3x } \end{array}$$

← $3x \cdot (x^2 + 1) = 3x^3 + 3x$

↑ Align the $3x$ term under the x to make the next step easier.

Step 3: Subtract and bring down the remaining terms.

$$\begin{array}{r l} & 3x \\ x^2 + 1 \big) & \overline{3x^3 + 4x^2 + x + 7} \\ & \underline{3x^3 + 3x } \\ & 4x^2 - 2x + 7 \end{array}$$

← Subtract (change the signs and add).

← Bring down the $4x^2$ and the 7.

Solution continued

Step 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$\begin{array}{r|l} & 3x + 4 \\ \hline x^2 + 1 & 3x^3 + 4x^2 + x + 7 \\ & 3x^3 + 3x \\ \hline & 4x^2 - 2x + 7 \\ & 4x^2 + 4 \\ \hline & -2x + 3 \end{array}$$

← Divide $4x^2$ by x^2 to get 4.

← Multiply $x^2 + 1$ by 4; subtract.

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

✓**Check:** (Quotient) (Divisor) + Remainder

$$= (3x + 4)(x^2 + 1) + (-2x + 3)$$

$$= 3x^3 + 3x + 4x^2 + 4 + (-2x + 3)$$

$$= 3x^3 + 4x^2 + x + 7 = \text{Dividend}$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

Theorem

Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

Work with Polynomials in Two Variables

Example

Using a Special Product Formula

To multiply $(2x - y)^2$, use the Squares of Binomials formula (3b) with $2x$ instead of x and with y instead of a.

$$\begin{aligned}(2x - y)^2 &= (2x)^2 - 2 \cdot y \cdot 2x + y^2 \\ &= 4x^2 - 4xy + y^2\end{aligned}$$