1 Functions

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Moroccan runner Hicham El Guerrouj, current world record holder for the mile run, bested the record set 6 years earlier by 1.26 seconds.

1.2 EXPONENTS

Introduction

Introduction

- Not all variables are related linearly. In this section we will discuss exponents, which will enable us to express many *nonlinear* relationships.

Positive Integer Exponents

Positive Integer Exponents

- Numbers may be expressed with exponents, as in

  $2^3 = 2 \cdot 2 \cdot 2 = 8$.

  More generally, for any positive integer n, x^n means the product of n x's.

$$x^n = \overbrace{x \cdot x \cdot \cdots \cdot x}^{n}$$

- The number being raised to the power is called the **base** and the power is the **exponent:**

Positive Integer Exponents

There are several *properties of exponents* for simplifying expressions. The first three are known, respectively, as the addition, subtraction, and multiplication properties of exponents.

Properties of Exponents

Property	Description	Brief Examples
$x^m \cdot x^n = x^{m+n}$	To *multiply* powers of the same base, *add* the exponents	$x^2 \cdot x^3 = x^5$
$\dfrac{x^m}{x^n} = x^{m-n}$	To *divide* powers of the same base, *subtract* the exponents (top exponent minus bottom exponent)	$\dfrac{x^5}{x^3} = x^2$
$(x^m)^n = x^{m \cdot n}$	To raise a power to a power, *multiply* the powers	$(x^2)^3 = x^6$
$(xy)^n = x^n \cdot y^n$	To raise a product to a power, raise *each factor* to the power	$(2x)^3 = 2^3 \cdot x^3 = 8x^3$
$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	To raise a fraction to a power, raise the numerator *and* denominator to the power	$\left(\dfrac{x}{5}\right)^3 = \dfrac{x^3}{5^3} = \dfrac{x^3}{125}$

Zero and Negative Exponents

Zero and Negative Exponents

Any number except zero can be raised to a negative integer or zero power.

Zero and Negative Integer Exponents

For $x \neq 0$

		Brief Examples
$x^0 = 1$	x to the power 0 is 1	$5^0 = 1$
$x^{-1} = \frac{1}{x}$	x to the power -1 is one over x	$7^{-1} = \frac{1}{7}$
$x^{-2} = \frac{1}{x^2}$	x to the power -2 is one over x squared	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
$x^{-n} = \frac{1}{x^n}$	x to a negative power is one over x to the positive power	$(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

Note that 0^0 and 0^{-3} are *undefined*.

Zero and Negative Exponents

The definitions of x^0 and x^{-n} are motivated by the following calculations.

$$1 = \frac{x^2}{x^2} = x^{2-2} = x^0$$

The subtraction property of exponents leads to $x^0 = 1$

$$\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n}$$

$x^0 = 1$ and the subtraction property of exponents lead to $x^{-n} = \frac{1}{x^n}$

Zero and Negative Exponents

A fraction to a negative power means *division* by the fraction, so we "invert and multiply."

$$\left(\frac{x}{y}\right)^{-1} = \frac{1}{\frac{x}{y}} = 1 \cdot \frac{y}{x} = \frac{y}{x}$$

Reciprocal of the original fraction

Therefore, for and $x \neq 0$ and $y \neq 0$,

$\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$	A fraction to the power -1 is the reciprocal of the fraction
$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$	A fraction to the negative power is the reciprocal of the fraction to the positive power

Example 1 – *SIMPLIFYING FRACTIONS TO NEGATIVE EXPONENTS*

a. $$\left(\frac{3}{2}\right)^{-1} = \frac{2}{3}$$

Reciprocal of the original

b. $$\left(\frac{1}{2}\right)^{-3} = \left(\frac{2}{1}\right)^{3} = \frac{2^3}{1^3} = 8$$

Roots and Fractional Exponents

Roots and Fractional Exponents

We may take the square root of any *nonnegative* number, and the cube root of *any* number.

Example 2 – *EVALUATING ROOTS*

a. $\sqrt{9} = 3$

b. $\sqrt{-9}$ is undefined.

Square roots of negative numbers are not defined

c. $\sqrt[3]{8} = 2$

d. $\sqrt[3]{-8} = -2$

Cube roots of negative numbers *are* defined

e. $\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$

Roots and Fractional Exponents

There are *two* square roots of 9, namely 3 and –3, but the radical sign $\sqrt{\ }$ means just the *positive* one (the "principal" square root).

$\sqrt[n]{a}$ means the principal nth root of a.

Principal means the positive root if there are two

In general, we may take *odd* roots of *any* number, but *even* roots only if the number is positive or zero.

Fractional Exponents

Fractional Exponents

Fractional exponents are defined as follows:

Powers of the Form $\frac{1}{n}$		Brief Examples
$x^{\frac{1}{2}} = \sqrt{x}$	Power $\frac{1}{2}$ means the principal square root	$9^{\frac{1}{2}} = \sqrt{9} = 3$
$x^{\frac{1}{3}} = \sqrt[3]{x}$	Power $\frac{1}{3}$ means the cube root	$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$
$x^{\frac{1}{n}} = \sqrt[n]{x}$	Power $\frac{1}{n}$ means the principal nth root (for a positive integer n)	$(-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$

Fractional Exponents

The definition of $x^{\frac{1}{2}}$ is motivated by the multiplication property of exponents:

$$\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2} \cdot 2} = x^1 = x$$

Taking square roots of each side of $\left(x^{\frac{1}{2}}\right)^2 = x$ es

$$x^{\frac{1}{2}} = \sqrt{x}$$

x to the half power means the square root of x

Example 4 – *EVALUATING FRACTIONAL EXPONENTS*

a. $\left(\frac{4}{25}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$

b. $\left(-\frac{27}{8}\right)^{\frac{1}{3}} = \sqrt[3]{-\frac{27}{8}} = -\frac{\sqrt[3]{27}}{\sqrt[3]{8}} = -\frac{3}{2}$

Fractional Exponents

To define $x^{\frac{m}{n}}$ for positive integers *m* and *n*, the exponent $\frac{m}{n}$ must be fully reduced (for example, $\frac{4}{6}$ must be reduced to $\frac{2}{3}$). Then

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(x^m\right)^{\frac{1}{n}}$$

Since in both cases the exponents multiply to $\frac{m}{n}$

Therefore, we define:

Fractional Exponents

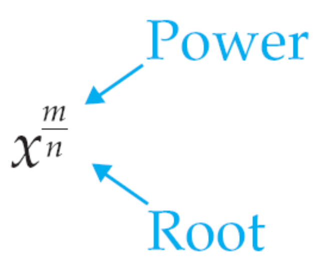
$$x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$$

$x^{m/n}$ means the mth power of the nth root, or equivalently, the nth root of the mth power

Fractional Exponents

Both expressions, $\left(\sqrt[n]{x}\right)^m$ and $\sqrt[n]{x^m}$, will give the same answer.

In either case *the numerator determines the power* and *the denominator determines the root*.



Power over root

Fractional Exponents

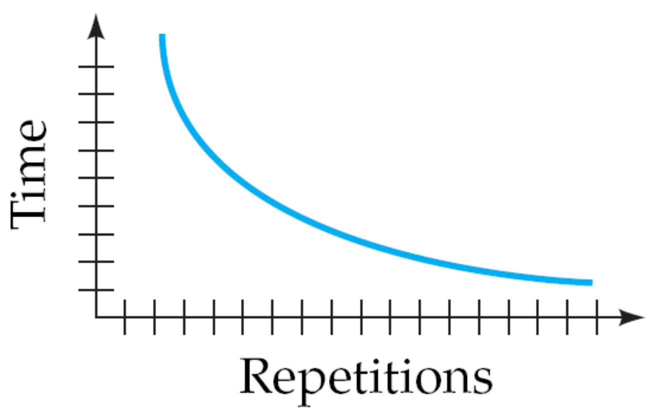
$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^2$ is the first number squared plus twice the product of the numbers plus the second number squared

Learning Curves in Airplane Production

Learning Curves in Airplane Production

It is a truism that the more you practice a task, the faster you can do it. Successive repetitions generally take less time, following a “learning curve” like that given below. Learning curves are used in industrial production.



Learning Curves in Airplane Production

For example, it took 150,000 work-hours to build the first Boeing 707 airliner, while later planes (n = 2, 3, ... , 300) took less time.

$$\begin{pmatrix}\text{Time to build} \\ \text{plane number } n\end{pmatrix} = 150\, n^{-0.322} \text{ thousand work-hours}$$

The time for the 10th Boeing 707 is found by substituting

n = 10:

$$\begin{pmatrix}\text{Time to build} \\ \text{plane 10}\end{pmatrix} = 150(10)^{-0.322} \qquad 150n^{-0.322} \text{ with } n = 10$$

$$\approx 71.46 \text{ thousand work-hours} \qquad \text{Using a calculator}$$

Learning Curves in Airplane Production

This shows that building the 10th Boeing 707 took about 71,460 work-hours, which is less than half of the 150,000 work-hours needed for the first. For the 100th 707:

$$\begin{pmatrix}\text{Time to build} \\ \text{plane } 100\end{pmatrix} = 150(100)^{-0.322} \qquad 150n^{-0.322} \text{ with } n = 100$$

$$\approx 34.05 \text{ thousand work-hours}$$

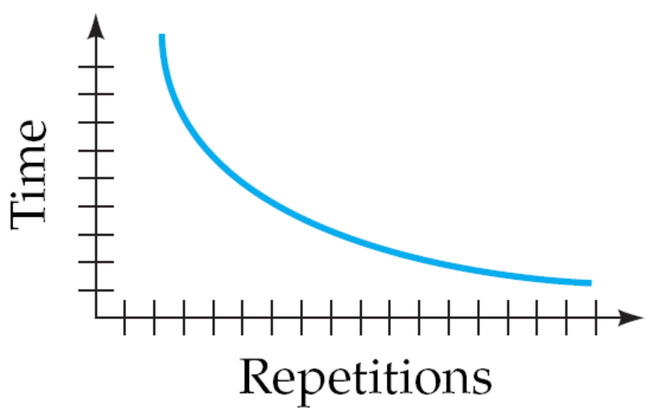
or about 34,050 work-hours, which is less than the half time needed to build the 10th.

Such learning curves are used for determining the cost of a contract to build several planes.

Learning Curves in Airplane Production

The learning curve graphed shown below decreases less steeply as the number of repetitions increases.

This means that while construction time continues to decrease, it does so more slowly for later planes. This behavior, called **diminishing returns,** is typical of learning curves.



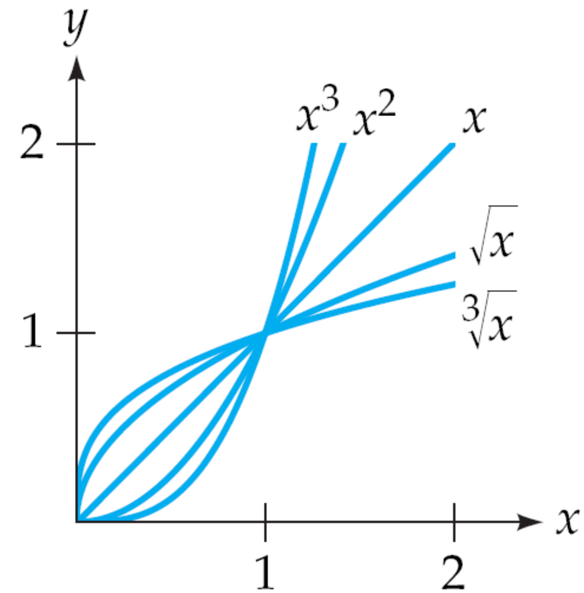
Power Regression (Optional)

Power Regression (Optional)

Just as we used *linear* regression to fit a *line* to data points, we can use **power regression** to fit a *power curve* to data points. The procedure is easily accomplished using a graphing calculator (or spreadsheet or other computer software).

When do you use power regression instead of linear regression (or some other type)?

You should look at a graph of the data and see if it lies more along a *curve* as shown in the image rather than along a line.



Power Regression (Optional)

Furthermore, sometimes there are *theoretical* reasons to prefer a curve.

For example, sales of a product may increase linearly for a short time, but then usually grow more slowly because of market saturation or competition, and so are best modeled by a curve.