# 1 Functions

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Moroccan runner Hicham El Guerrouj, current world record holder for the mile run, bested the record set 6 years earlier by 1.26 seconds.

# 1.1 REAL NUMBERS, INEQUALITIES, AND LINES

# Introduction

# Introduction

- Quite simply, *calculus is the study of rates of change*. We will use calculus to analyze rates of inflation, rates of learning, rates of population growth, and rates of natural resource consumption.

- In this first section we will study **linear** relationships between two variable quantities—that is, relationships that can be represented by **lines.**

# Real Numbers and Inequalities

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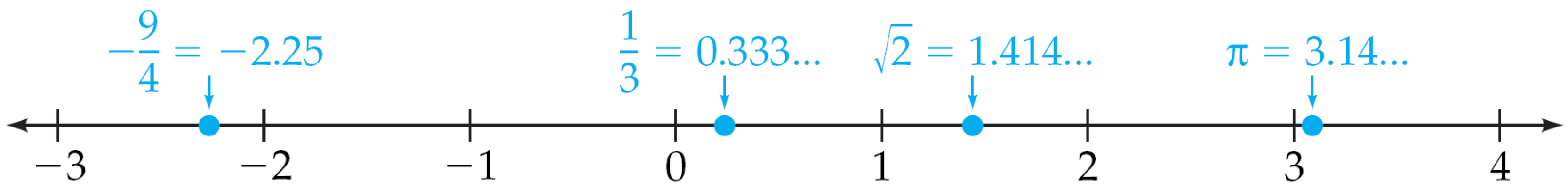
The word "number" means **real number,** a number that can be represented by a point on the number line (also called the **real line**).



# Real Numbers and Inequalities

- The *order* of the real numbers is expressed by **inequalities.** For example, $a < b$ means "$a$ is to the *left* of $b$" or, equivalently, "$b$ is to the *right* of $a$."

**Inequalities**

| Inequality | In Words | Brief Examples |
|---|---|---|
| $a < b$ | $a$ is less than (smaller than) $b$ | $3 < 5$ |
| $a \leq b$ | $a$ is less than or equal to $b$ | $-5 \leq -3$ |
| $a > b$ | $a$ is greater than (larger than) $b$ | $\pi > 3$ |
| $a \geq b$ | $a$ is greater than or equal to $b$ | $2 \geq 2$ |

- The inequalities $a < b$ and $a > b$ are called **strict inequalities,** and $a \leq b$ and $a \geq b$ are called **nonstrict inequalities.**

# Real Numbers and Inequalities

A **double inequality,** such as $a < x < b$, means that *both* the inequalities $a < x$ and $x < b$ hold.

The inequality $a < x < b$ can be interpreted graphically as "$x$ is between $a$ and $b$."

# Sets and Intervals

# Sets and Intervals

**Braces** { } are read "the set of all" and a **vertical bar** | is read "such that."

# Example 1 – *INTERPRETING SETS*

The set of all

**a.** $\{x \mid x > 3\}$ means "the set of all $x$ such that $x$ is greater than 3."

Such that

**b.** $\{x \mid -2 < x < 5\}$ means "the set of all $x$ such that $x$ is between $-2$ and 5."

# Sets and Intervals

- The set $\{ x \mid 2 \leq x \leq 5 \}$ can be expressed in **interval notation** by enclosing the endpoints 2 and 5 in **square brackets,** [2, 5], to indicate that the endpoints are *included.*

- The set $\{ x \mid 2 \leq x \leq 5 \}$ can be written with **parentheses,** (2, 5), to indicate that the endpoints 2 and 5 are *excluded.*

- An interval is **closed** if it includes both endpoints, and **open** if it includes neither endpoint.

# Sets and Intervals

The four types of intervals are shown below:

a **solid dot** ● on the graph indicates that the point is *included* in the interval; a **hollow dot** ○ indicates that the point is *excluded.*

**Finite Intervals**

| *Interval Notation* | *Set Notation* | *Graph* | *Type* |
|---|---|---|---|
| $[a, b]$ | $\{x \mid a \le x \le b\}$ |  | Closed (includes endpoints) |
| $(a, b)$ | $\{x \mid a < x < b\}$ |  | Open (excludes endpoints) |
| $[a, b)$ | $\{x \mid a \le x < b\}$ |  | Half-open or half-closed |
| $(a, b]$ | $\{x \mid a < x \le b\}$ |  | Half-open or half-closed |

*Brief Examples*

| | |
|---|---|
| $[-2, 5]$ |  |
| $(-2, 5)$ |  |
| $[-2, 5)$ |  |
| $(-2, 5]$ |  |

# Sets and Intervals

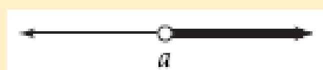
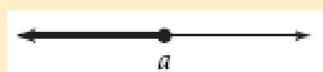
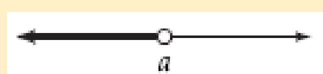
- An interval may extend infinitely far to the *right* (indicated by the symbol $\infty$ for **infinity**) or infinitely far to the *left* (indicated by $-\infty$ for **negative infinity**).

- Note that $\infty$ and $-\infty$ are not numbers, but are merely symbols to indicate that the interval extends endlessly in that direction.

# Sets and Intervals

The infinite intervals shown below are said to be **closed** or **open** depending on whether they *include* or *exclude* their single endpoint.

**Infinite Intervals**

| *Interval Notation* | *Set Notation* | *Graph* | *Type* | *Brief Examples* | |
|---|---|---|---|---|---|
| $[a, \infty)$ | $\{x \mid x \geq a\}$ |  | Closed | $[3, \infty)$ |  |
| $(a, \infty)$ | $\{x \mid x > a\}$ |  | Open | $(3, \infty)$ |  |
| $(-\infty, a]$ | $\{x \mid x \leq a\}$ |  | Closed | $(-\infty, 5]$ |  |
| $(-\infty, a)$ | $\{x \mid x < a\}$ |  | Open | $(-\infty, 5)$ |  |

# Sets and Intervals

We use *parentheses* rather than square brackets with $\infty$ and $-\infty$ since they are not actual numbers.

The interval $(-\infty, \infty)$ extends infinitely far in *both* directions (meaning the entire real line) and is also denoted by $\mathbb{R}$ (the set of all real numbers).

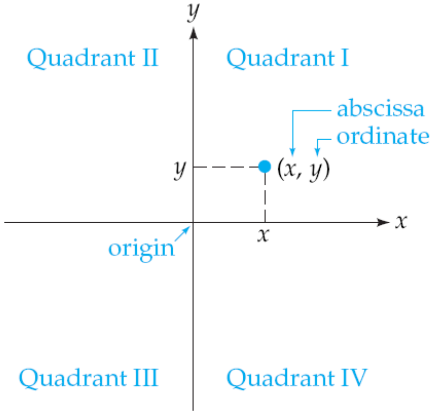
$$\mathbb{R} = (-\infty, \infty)$$

# Cartesian Plane

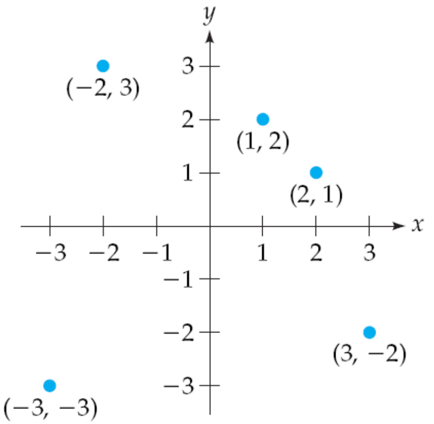
# Cartesian Plane

- Two real lines or **axes,** one horizontal and one vertical, intersecting at their zero points, define the **Cartesian plane.**

- The point where they meet is called the **origin.** The axes divide the plane into four **quadrants,** I through IV, as shown below.




The Cartesian plane




The Cartesian plane with several points. Order matters: (1, 2) is not the same as (2, 1)

# Cartesian Plane

Any point in the Cartesian plane can be specified uniquely by an ordered pair of numbers ($x$, $y$); $x$, called the **abscissa** or ***x*-coordinate,** is the number on the horizontal axis corresponding to the point; $y$, called the **ordinate** or ***y*-coordinate,** is the number on the vertical axis corresponding to the point.

# Lines and Slopes

# Lines and Slopes

The symbol Δ (read "delta," the Greek letter D) means "the change in."

For any two points ($x_1$, $y_1$) and ($x_2$ , $y_2$) we define

$\Delta x = x_2 - x_1$ The change in $x$ is the difference in the $x$-coordinates

$\Delta y = y_2 - y_1$ The change in $y$ is the difference in the $y$-coordinates

# Lines and Slopes

Any two distinct points determine a line. A nonvertical line has a **slope** that measures the *steepness* of the line, and is defined as *the change in y divided by the change in x* for any two points on the line.

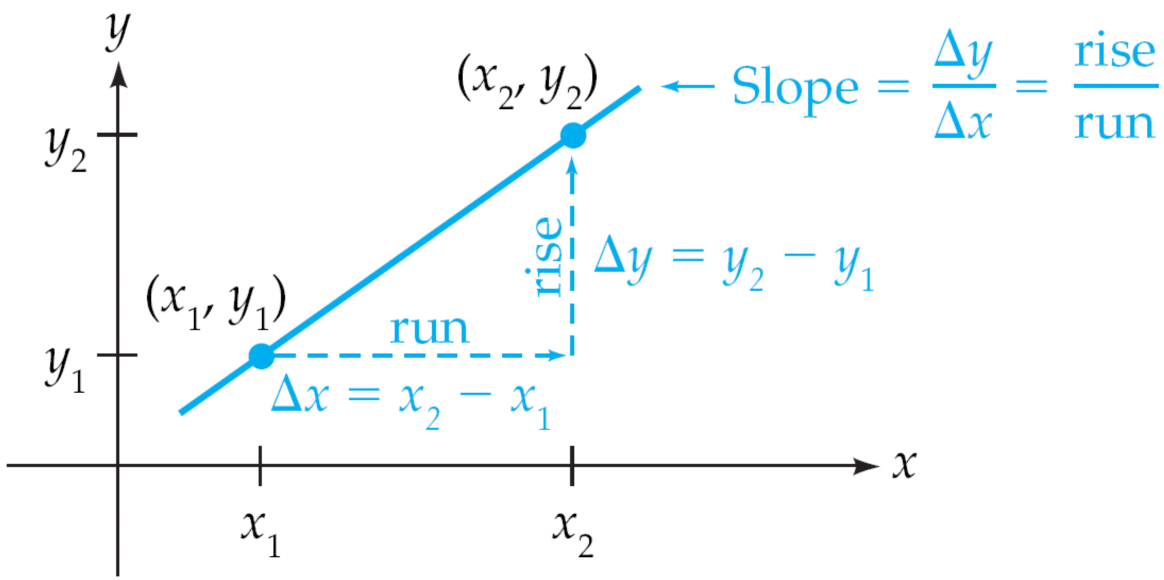
**Slope of Line Through $(x_1, y_1)$ and $(x_2, y_2)$**

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope is the change in $y$ over the change in $x$ $(x_2 \neq x_1)$

# Lines and Slopes

The changes Δ*y* and Δ*x* are often called, respectively, the "rise" and the "run," with the understanding that a negative "rise" means a "fall." Slope is then "rise over run."



# Example 2 – *FINDING SLOPES AND GRAPHING LINES*

Find the slope of the line through each pair of points, and graph the line.

**a.** (2, 1), (3, 4) **b.** (2, 4), (3, 1)

**c.** (–1, 3), (2, 3) **d.** (2, –1), (2, 3)

Solution:

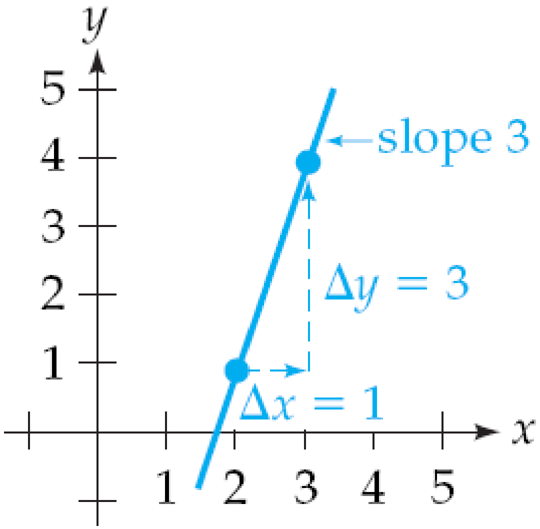
We use the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for each pair $(x_1, y_1)$, $(x_2, y_2)$.

# Example 2 – *Solution*

cont'd

**a.** For (2, 1) and (3, 4) the slope is

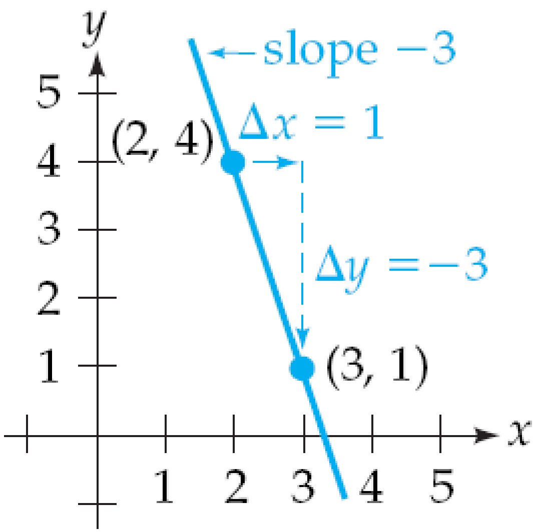
$$\frac{4 - 1}{3 - 2} = \frac{3}{1} = 3.$$



# Example 2 – *Solution*

cont'd

**b.** For (2, 4) and (3, 1) the slope is $\dfrac{1-4}{3-2} = \dfrac{-3}{1} = -3$.



# Example 2 – *Solution*

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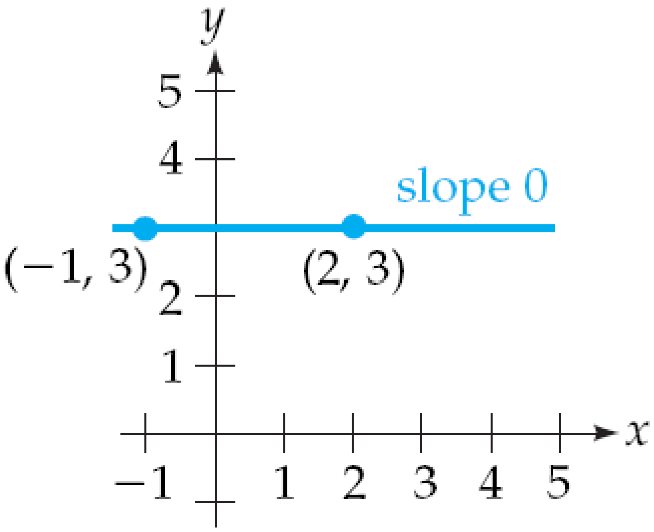
**c.** For $(-1, 3)$ and $(2, 3)$ the slope is

$$\frac{3-3}{2-(-1)} = \frac{0}{3} = 0.$$



# Example 2 – *Solution*

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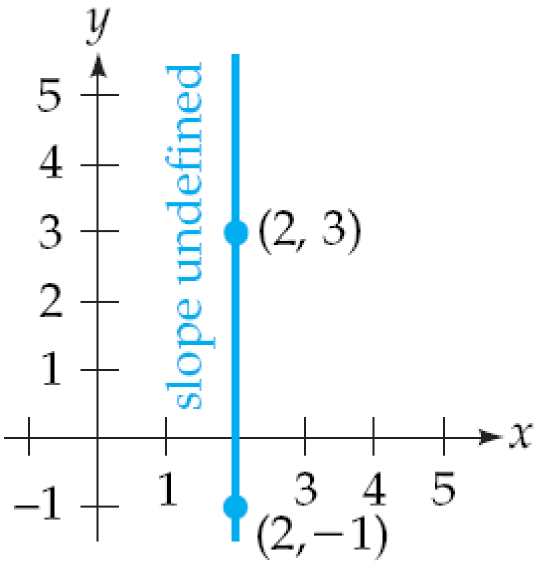
**d.** For (2, – 1) and (2, 3) the slope is *undefined:*

$$\frac{3 - (-1)}{2 - 2} = \frac{4}{0}.$$



# Lines and Slopes

When the *x*-coordinates are the same [as in Example 2(d)], the line is *vertical*, and when the *y*-coordinates are the same [as in Example 2(c)], the line is *horizontal*.

If $\Delta x = 1$, as in Examples 2(a) and 2(b), then the slope is just the "rise," giving an alternative definition for slope:

$$\text{Slope} = \left(\begin{array}{c}\text{Amount that the line rises}\\ \text{when } x \text{ increases by } 1\end{array}\right)$$

# Equations of Lines

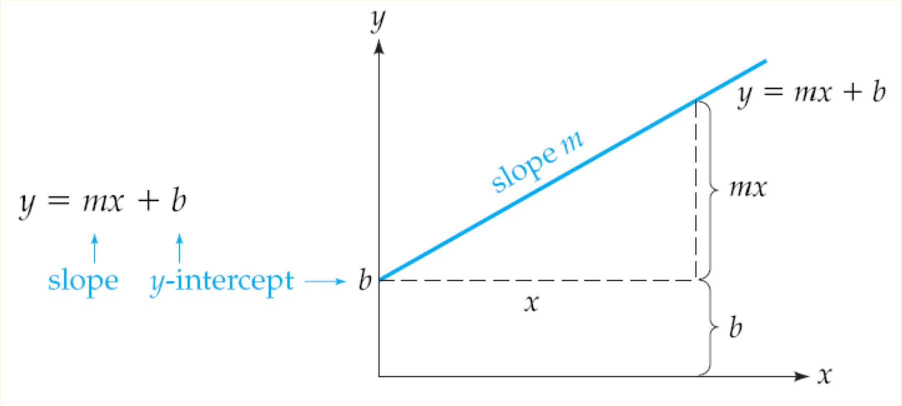
# Equations of Lines

- The point where a nonvertical line crosses the *y*-axis is called the ***y*-intercept** of the line.

- The *y*-intercept can be given either as the *y*-coordinate *b* or as the point (0, *b*). Such a line can be expressed very simply in terms of its slope and *y*-intercept, representing the points by variable coordinates (or “variables”) *x* and *y*.

# Equations of Lines

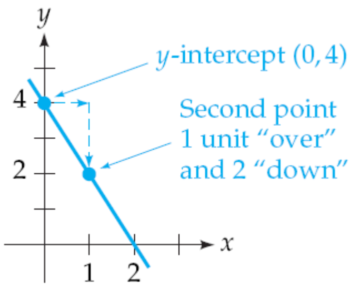
**Slope-Intercept Form of a Line**




*Brief Example*

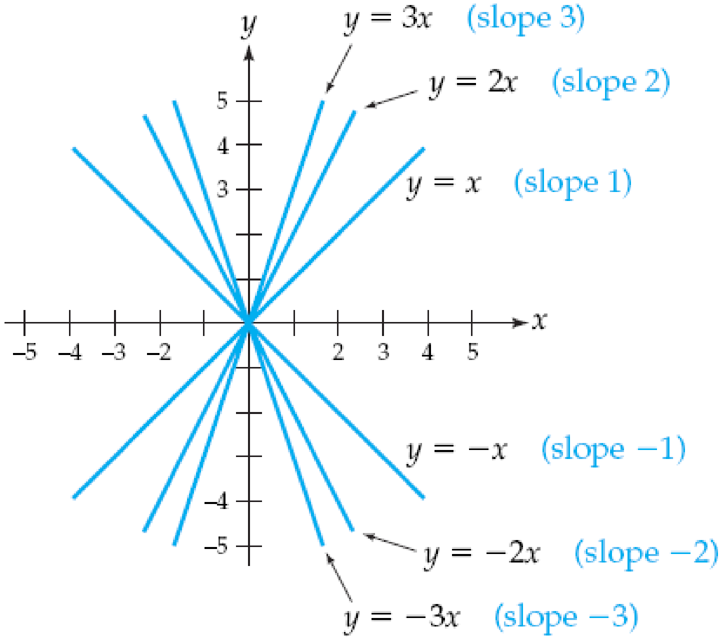
For the line with slope −2 and $y$-intercept 4:

$$y = -2x + 4$$



# Equations of Lines

For lines through the origin, the equation takes the particularly simple form, $y = mx$ (since $b = 0$), as illustrated below.



# Equations of Lines

The most useful equation for a line is the *point-slope form*.

**Point-Slope Form of a Line**

$$y - y_1 = m(x - x_1)$$

$(x_1, y_1)$ = point on the line
$m$ = slope

This form comes directly from the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ by replacing $x_2$ and $y_2$ by $x$ and $y$, and then multiplying each side by $(x - x_1)$.

It is most useful when you know the slope of the line and a point on it.

# Example 3 – *USING THE POINT-SLOPE FORM*

Find an equation of the line through (6, –2) with slope $-\frac{1}{2}$.

Solution:

$$y - (-2) = -\frac{1}{2}(x - 6) \qquad y - y_1 = m(x - x_1) \text{ with } y_1 = -2,\ m = -\tfrac{1}{2}, \text{ and } x_1 = 6$$

$$y + 2 = -\frac{1}{2}x + 3 \qquad \text{Eliminating parentheses}$$

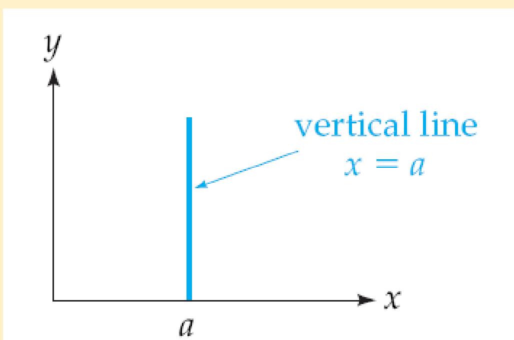
$$y = -\frac{1}{2}x + 1 \qquad \text{Subtracting 2 from each side}$$

# Equations of Lines

Vertical and horizontal lines have particularly simple equations: a variable equaling a constant.

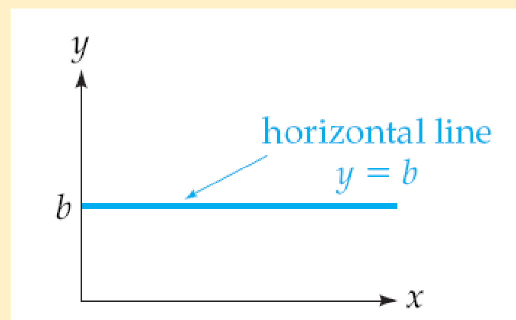
***Vertical Line***

$$x = a$$




***Horizontal Line***

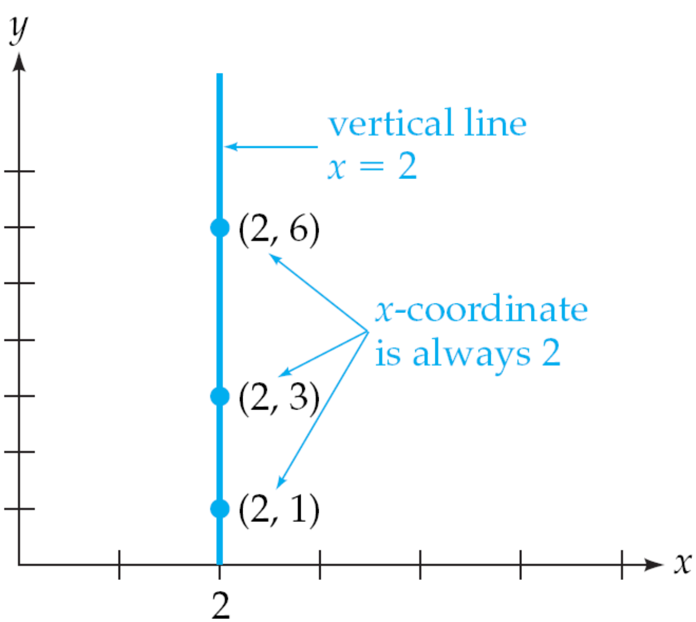
$$y = b$$

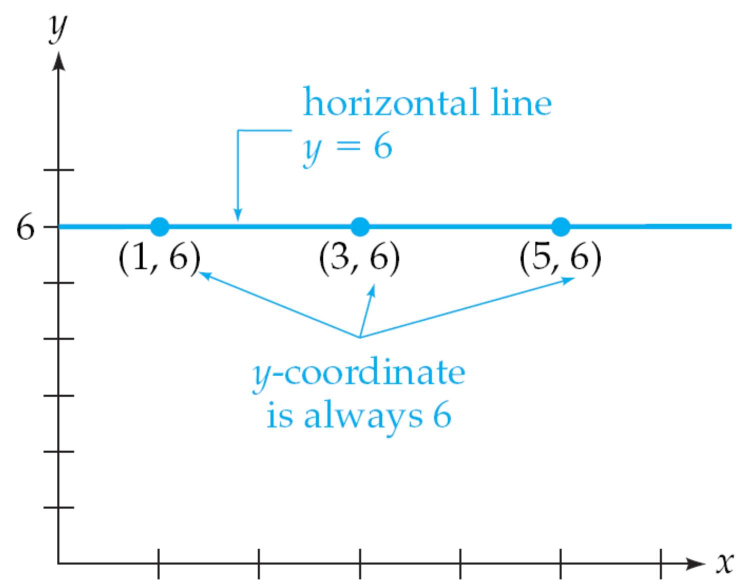


# Example 5 – *GRAPHING VERTICAL AND HORIZONTAL LINES*

Graph the lines $x = 2$ and $y = 6$.

Solution:



# Equations of Lines

In a vertical line, the *x*-coordinate does not change, so $\Delta x = 0$, making the slope $m = \Delta y/\Delta x$ *undefined*. Therefore, distinguish carefully between slopes of vertical and horizontal lines:

- Vertical line: slope is *undefined.*
- Horizontal line: slope *is* defined, and is *zero.*

# Equations of Lines

There is one form that covers *all* lines, vertical and nonvertical.

| General Linear Equation | |
|---|---|
| $ax + by = c$ | For constants $a$, $b$, $c$, with $a$ and $b$ not both zero |

Any equation that can be written in this form is called a **linear equation,** and the variables are said to **depend linearly** on each other.

## Example 7 – *FINDING THE SLOPE AND THE y-INTERCEPT FROM A LINEAR EQUATION*

Find the slope and *y*-intercept of the line $2x + 3y = 12$.

Solution:

We write the line in slope-intercept form.

Solving for *y*:

$$3y = -2x + 12$$

Subtracting $2x$ from both sides of $2x + 3y = 12$

$$y = -\frac{2}{3}x + 4$$

Dividing each side by 3 gives the slope-intercept form $y = mx + b$

Therefore, the slope is $-\frac{2}{3}$ and the *y*-intercept is (0, 4).

# Linear Regression (Optional)

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Some real-world situations involve *many* data points, which may lie approximately but not exactly on a line.

How can we find the line that, in some sense, *lies closest* to the points or *best approximates* the points?

The most widely used technique is called **linear regression** or **least squares.**

Even without studying its mathematical basis, however, we can easily find the regression line using a graphing calculator.