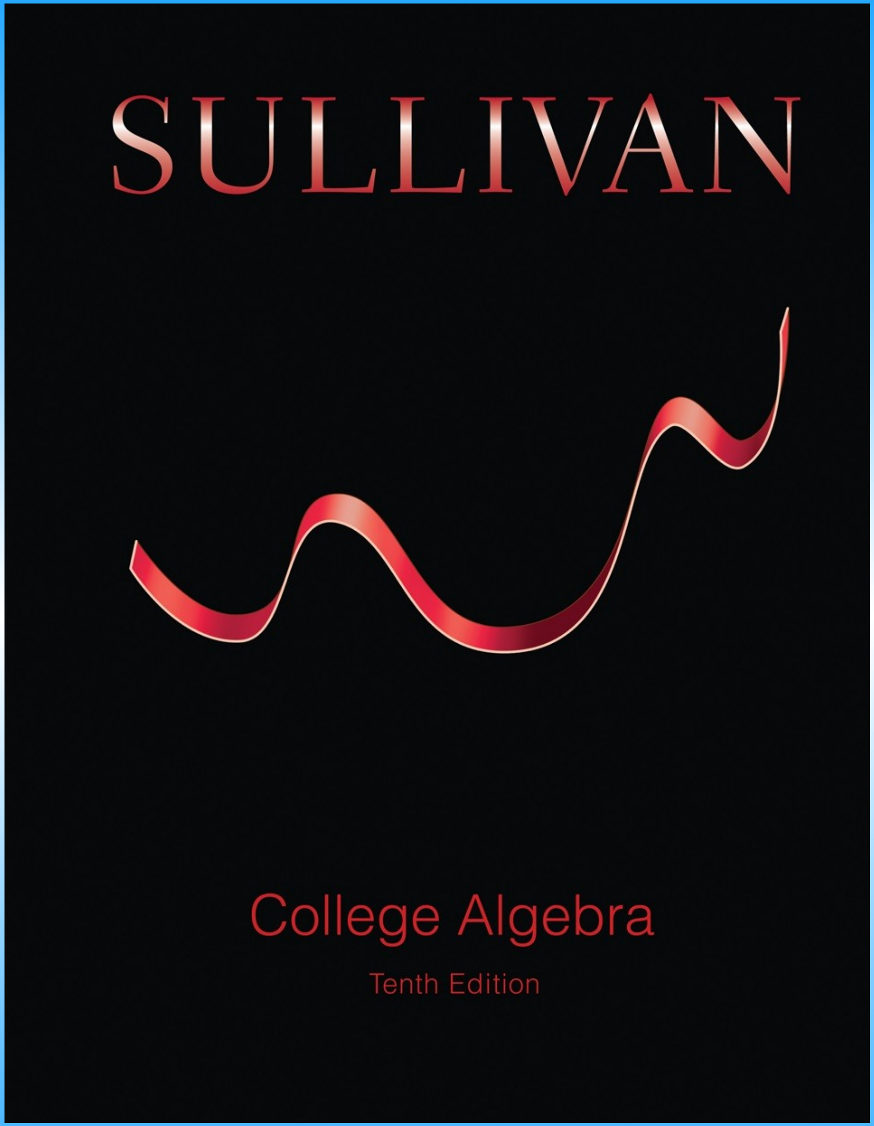
Chapter 5

Section 3



5.3 The Graph of a Rational Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

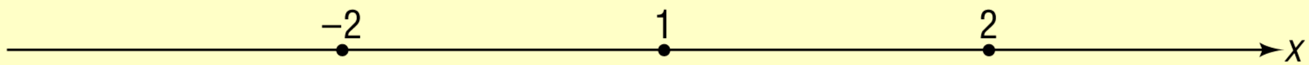
- Intercepts Section 2.2, pp. 159–160

Now Work the **'Are You Prepared?'** problem on page 365.

OBJECTIVES
1. Analyze the Graph of a Rational Function (p. 353)
2. Solve Applied Problems Involving Rational Functions (p. 364)

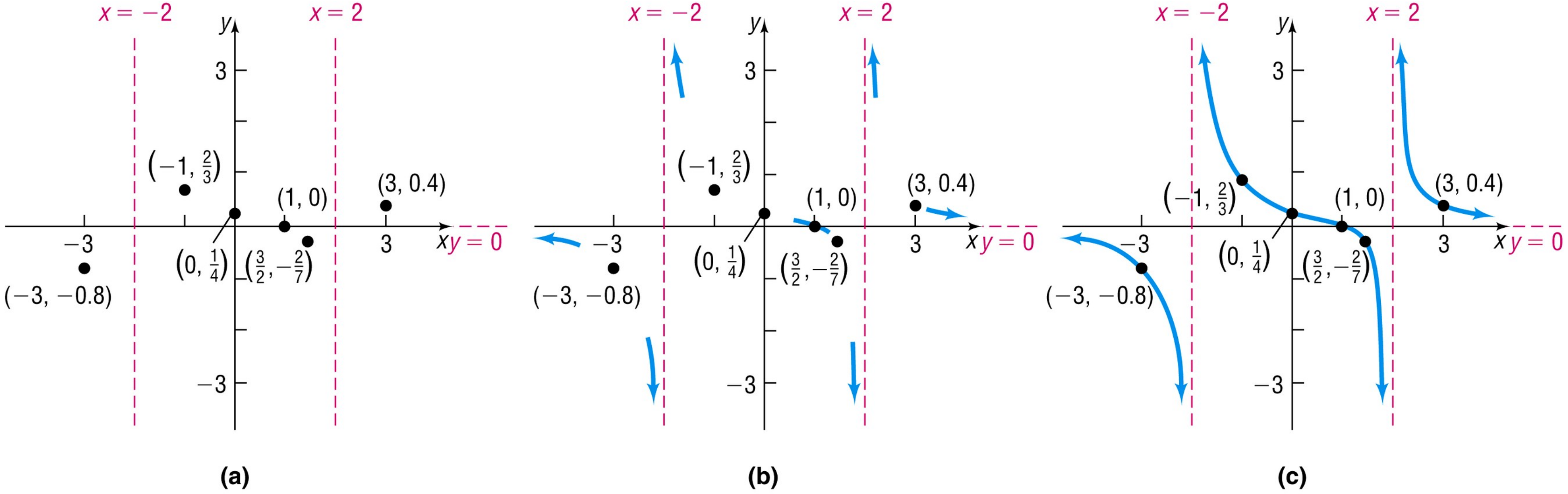
Analyze the Graph of a Rational Function

Table: Graphing $R(x) = \dfrac{x - 1}{x^2 - 4}$

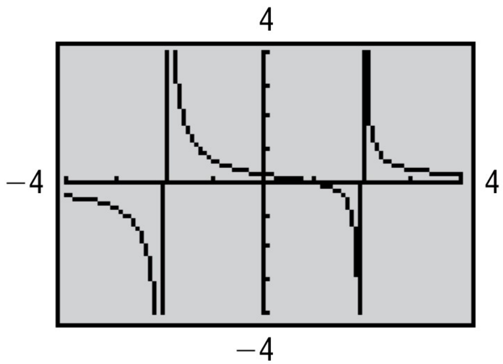


Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(2, \infty)$
Number chosen	-3	-1	$\frac{3}{2}$	3
Value of *R*	$R(-3) = -0.8$	$R(-1) = \frac{2}{3}$	$R\left(\frac{3}{2}\right) = -\frac{2}{7}$	$R(3) = 0.4$
Location of graph	Below *x*-axis	Above *x*-axis	Below *x*-axis	Above *x*-axis
Point on graph	$(-3, -0.8)$	$\left(-1, \frac{2}{3}\right)$	$\left(\frac{3}{2}, -\frac{2}{7}\right)$	$(3, 0.4)$

Figure

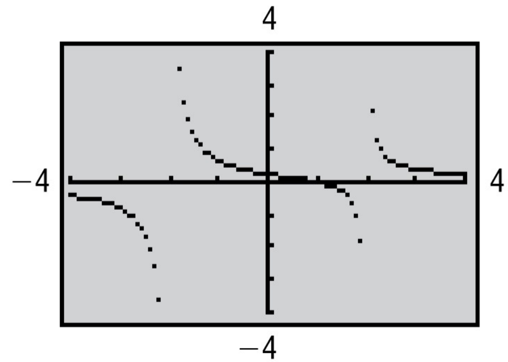


Figure



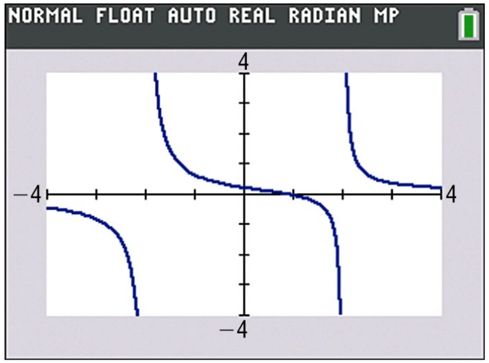
Connected mode with extraneous vertical lines

(a)



Dot mode

(b)



Connected mode without extraneous vertical lines

(c)

Example

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function:

$$R(x) = \frac{x^2 - 4}{x}$$

Solution

Step 1: $R(x) = \dfrac{x^2 - 4}{x} = \dfrac{(x+2)(x-2)}{x}$

The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: Because x cannot equal 0, there is no y-intercept. The graph has two x-intercepts, –2 and 2, each with odd multiplicity. Plot the points (–2, 0) and (2, 0). The graph will cross the x-axis at both points.

Solution continued

Step 4: The real zero of the denominator with R in lowest terms is 0, so the graph of R has the line $x = 0$ (the y-axis) as a vertical asymptote. Graph $x = 0$ using a dashed line.

The multiplicity of 0 is odd, so the graph will approach ∞ on one side of the asymptote $x = 0$, and $-\infty$ on the other side.

Solution continued

Step 5: Since the degree of the numerator, 2, is one greater than the degree of the denominator, 1, the rational function will have an oblique asymptote. To find the oblique asymptote, use long division.

$$\begin{array}{r} x \\ x\overline{)x^2 - 4} \\ \underline{x^2 \phantom{{}-4}} \\ -4 \end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. Graph $y = x$ using a dashed line.

Solution continued

To determine whether the graph of R intersects the asymptote $y = x$, solve the equation $R(x) = x$.

$$R(x) = \frac{x^2 - 4}{x} = x$$

$$x^2 - 4 = x^2$$

$$-4 = 0 \qquad \text{Impossible}$$

The equation $\frac{x^2 - 4}{x} = x$ has no solution, so the graph of R does not intersect the line $y = x$.

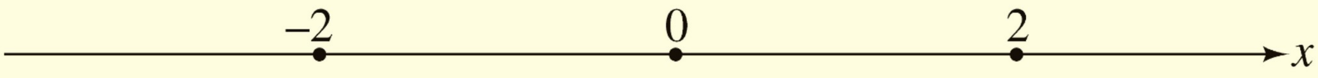
Solution continued

Step 6: The zeros of the numerator are –2 and 2; the zero of the denominator is 0. Use these values to divide the x-axis into four intervals:

$$(-\infty, -2) \qquad (-2, 0) \qquad (0, 2) \qquad (2, \infty)$$

Solution continued

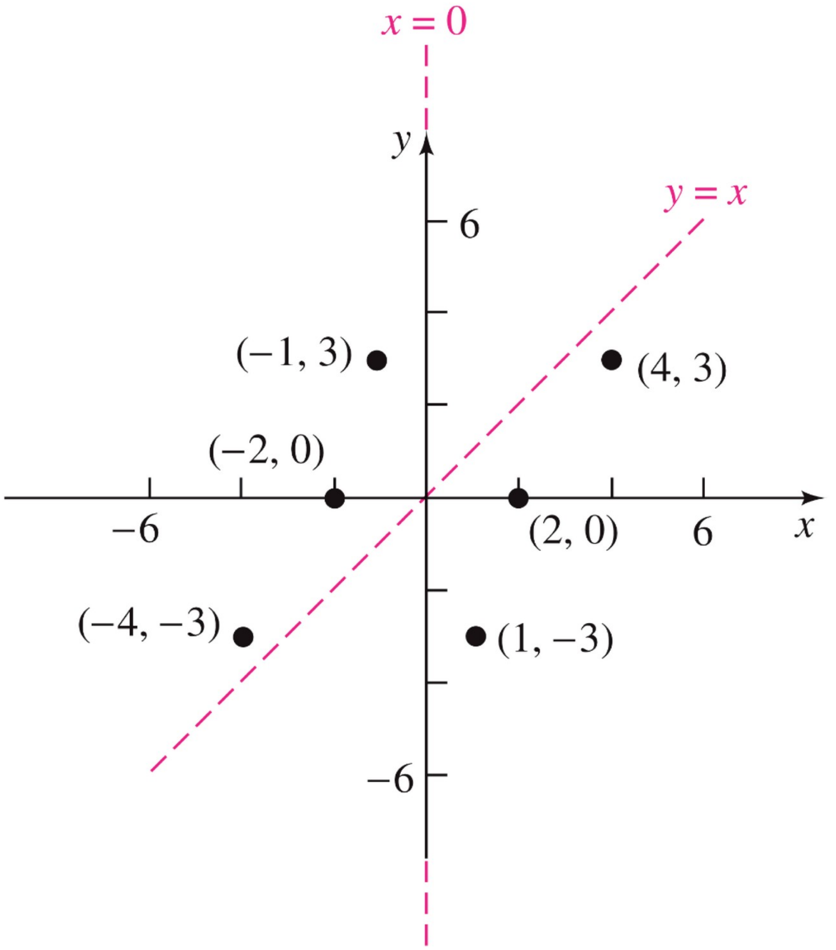
Now construct a table.



Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Number chosen	-4	-1	1	4
Value of R	$R(-4) = -3$	$R(-1) = 3$	$R(1) = -3$	$R(4) = 3$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-4, -3)$	$(-1, 3)$	$(1, -3)$	$(4, 3)$

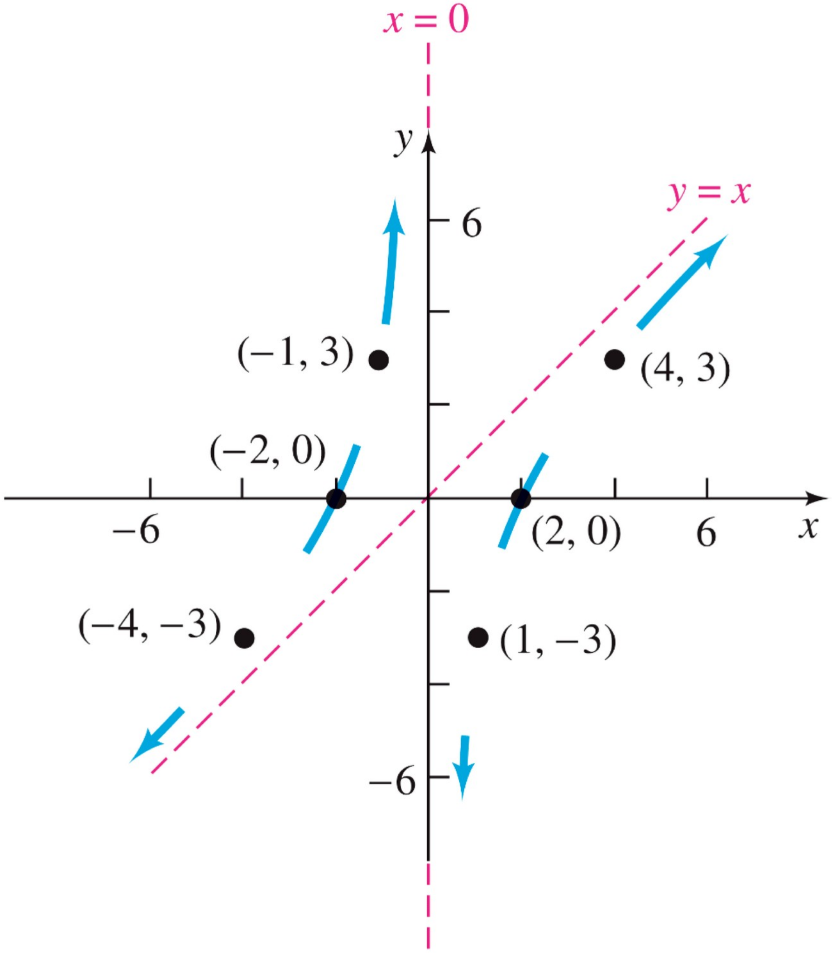
Solution continued

Step 7: Plot the points from the table along with the intercepts and asymptotes. The graph crosses the x-axis at $x = -2$ and $x = 2$, changing from being below the x-axis to being above it in both cases.



Solution continued

Because the graph of R is below the x-axis for $x < -2$ and is above the x-axis for $x > 2$, and because the graph of R does not intersect the oblique asymptote $y = x$, the graph of R will approach the line $y = x$ as shown in the figure.



Solution continued

Because the graph of R is above the x-axis for $-2 < x < 0$, the graph of R will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$.

$$\lim_{x \to 0^-} R(x) = \infty$$

Because the graph of R approaches ∞ on one side of the asymptote and $-\infty$ on the other, the graph of R will approach the vertical asymptote $x = 0$ at the bottom to the right of $x = 0$.

$$\lim_{x \to 0^+} R(x) = -\infty$$

Solution continued

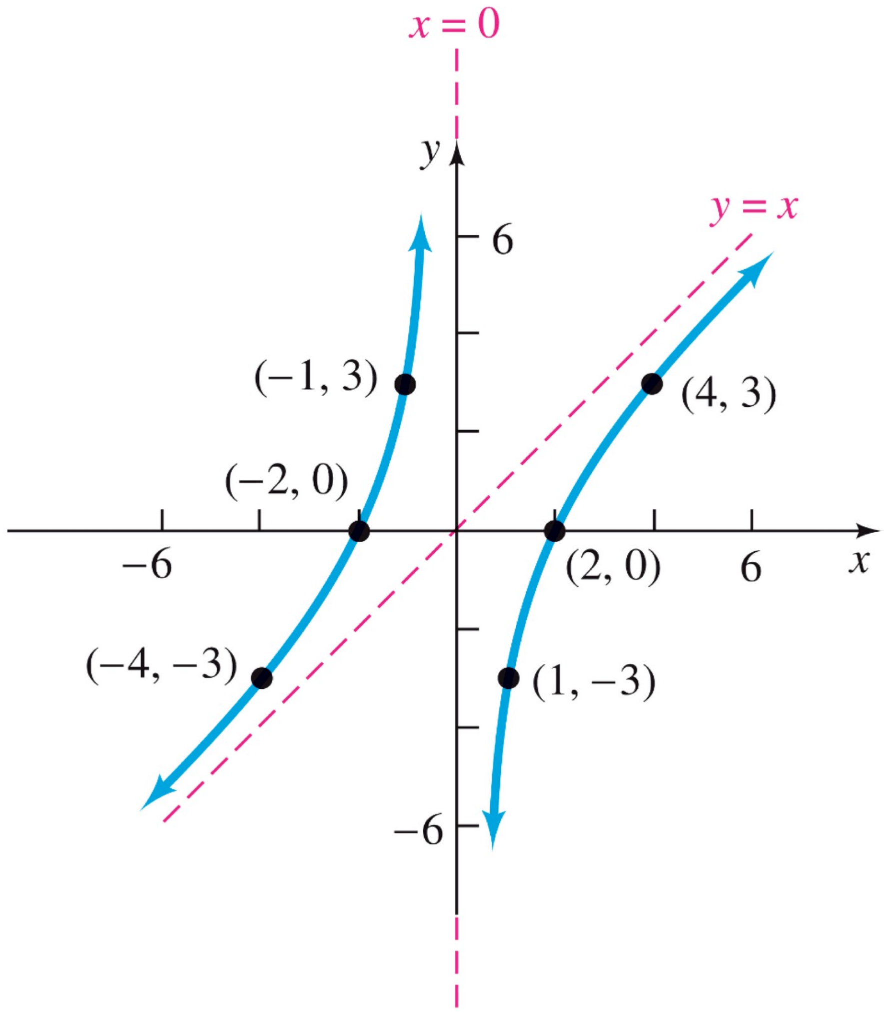
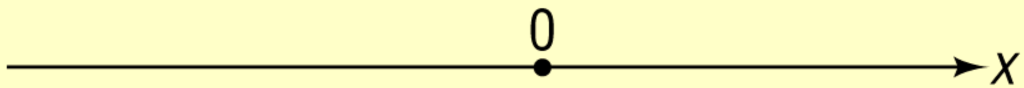
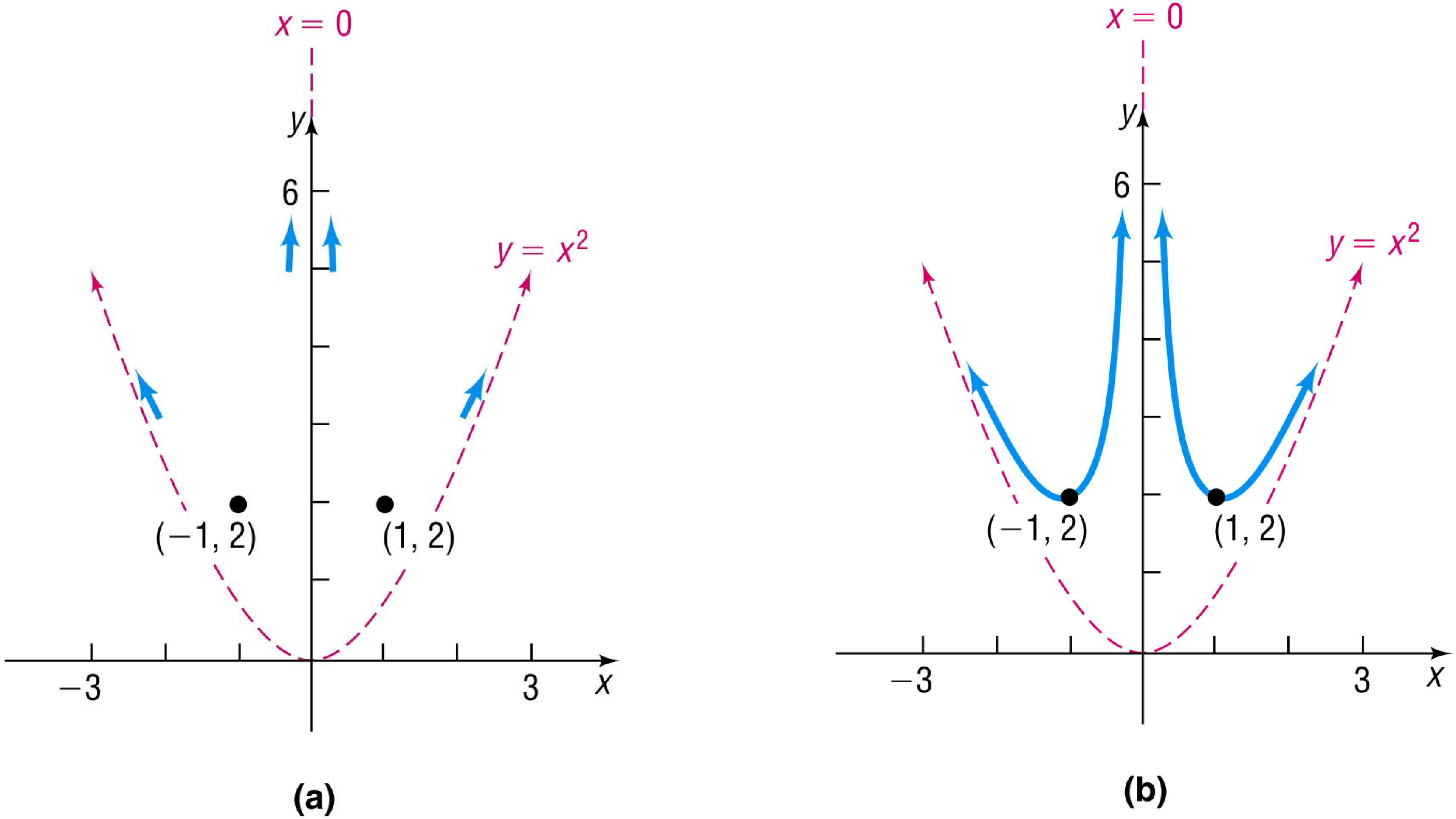


Table: Graphing $R(x) = \dfrac{x^4 + 1}{x^2}$



Interval	$(-\infty, 0)$	$(0, \infty)$
Number chosen	-1	1
Value of *R*	$R(-1) = 2$	$R(1) = 2$
Location of graph	Above x-axis	Above x-axis
Point on graph	$(-1, 2)$	$(1, 2)$

Figure



Example

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Solution

Step 1: Factor R to get

$$R(x) = \frac{3x(x-1)}{(x+4)(x-3)}$$

The domain of R is $\{x \mid x \neq -4, x \neq 3\}$.

Step 2: R is in lowest terms.

Step 3: The y-intercept is $R(0) = 0$. Plot the point $(0, 0)$. Since the real solutions of the equation $3x(x-1) = 0$ are $x = 0$ and $x = 1$, the graph has two x-intercepts, 0 and 1, each with odd multiplicity. Plot the points $(0, 0)$ and $(1, 0)$; the graph will cross the x-axis at both points.

Step 4: R is in lowest terms. The real solutions of the equation $(x+4)(x-3) = 0$ are $x = -4$ and $x = 3$, so the graph of R has two vertical asymptotes, the lines $x = -4$ and $x = 3$. Graph these lines using dashes. The multiplicities that give rise to the vertical asymptotes are both odd, so the graph will approach ∞ on one side of each vertical asymptote and $-\infty$ on the other side.

Solution continued

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$.

To find out whether the graph of R intersects the asymptote, solve the equation $R(x) = 3$.

$$\begin{aligned} R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} &= 3 \\ 3x^2 - 3x &= 3x^2 + 3x - 36 \\ -6x &= -36 \\ x &= 6 \end{aligned}$$

The graph intersects the line $y = 3$ at $x = 6$, and $(6, 3)$ is a point on the graph of R. Plot the point $(6, 3)$ and graph the line $y = 3$ using dashes.

Solution continued

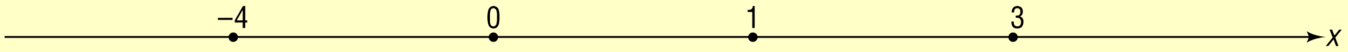
STEP 6: The real zeros of the numerator, 0 and 1, and the real zeros of the denominator, -4 and 3, divide the x-axis into five intervals:

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, 3) \quad (3, \infty)$$

Construct Table 12. See page 360. Plot the points from Table 12. Figure 34(a) shows the graph so far.

STEP 7: Since the graph of R is above the x-axis for $x < -4$ and only crosses the line $y = 3$ at $(6, 3)$, as x approaches $-\infty$ the graph of R will approach the horizontal asymptote $y = 3$ from above $\left(\lim_{x\to-\infty} R(x) = 3\right)$. The graph of R will approach the vertical asymptote $x = -4$ at the top to the left of $x = -4$ $\left(\lim_{x\to-4^-} R(x) = \infty\right)$ and at the bottom to the right of $x = -4$ $\left(\lim_{x\to-4^+} R(x) = -\infty\right)$. The graph of R will approach the vertical asymptote

Solution continued: Table 12



Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Number chosen	-5	-2	$\frac{1}{2}$	2	4
Value of *R*	$R(-5) = 11.25$	$R(-2) = -1.8$	$R\left(\frac{1}{2}\right) = \frac{1}{15}$	$R(2) = -1$	$R(4) = 4.5$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-5, 11.25)$	$(-2, -1.8)$	$\left(\frac{1}{2}, \frac{1}{15}\right)$	$(2, -1)$	$(4, 4.5)$

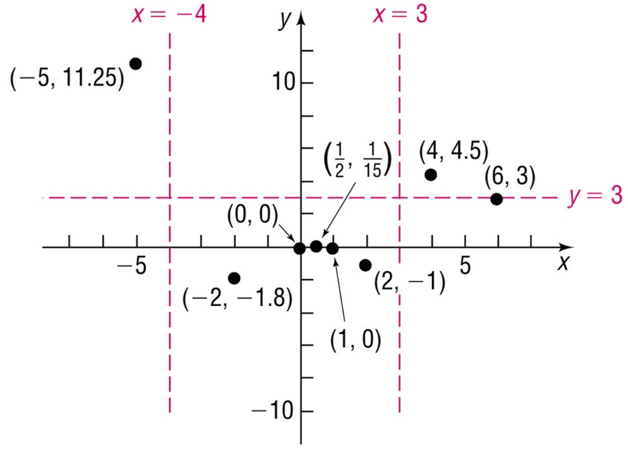
Solution continued

$x = 3$ at the bottom to the left of $x = 3$ $\left(\lim_{x \to 3^-} R(x) = -\infty\right)$ and at the top to the right of $x = 3$ $\left(\lim_{x \to 3^+} R(x) = \infty\right)$.

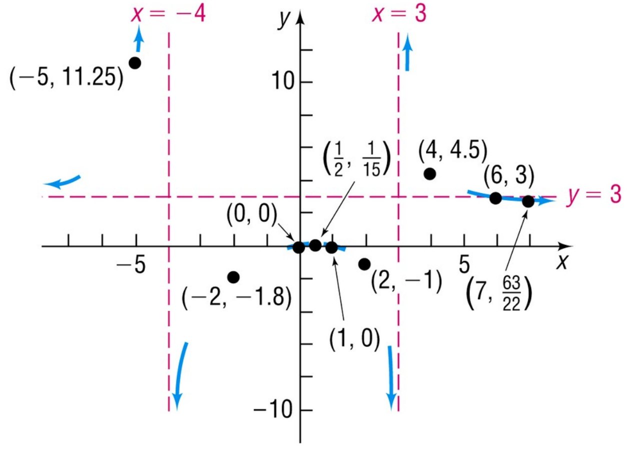
We do not know whether the graph of R crosses or touches the line $y = 3$ at $(6, 3)$. To see whether the graph, in fact, crosses or touches the line $y = 3$, plot an additional point to the right of $(6, 3)$. We use $x = 7$ to find $R(7) = \frac{63}{22} < 3$. The graph crosses $y = 3$ at $x = 6$. Because $(6, 3)$ is the only point where the graph of R intersects the asymptote $y = 3$, the graph must approach the line $y = 3$ from below as $x \to \infty$ $\left(\lim_{x \to \infty} R(x) = 3\right)$.

The graph crosses the x-axis at $x = 0$, changing from being below the x-axis to being above. The graph also crosses the x-axis at $x = 1$, changing from being above the x-axis to being below. See Figure 34(b). The complete graph is shown in Figure 34(c).

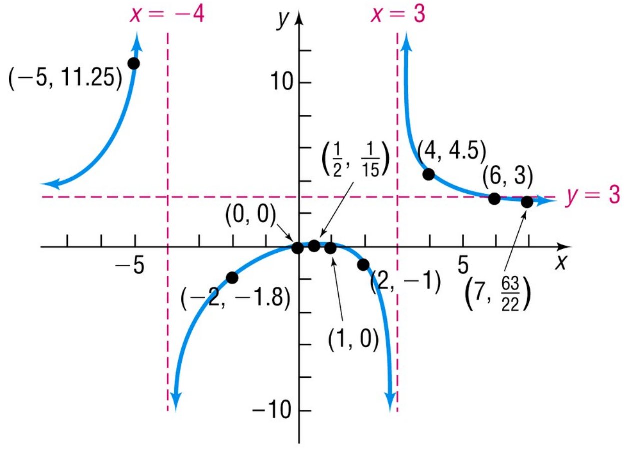
Solution continued: Figure 34



(a)



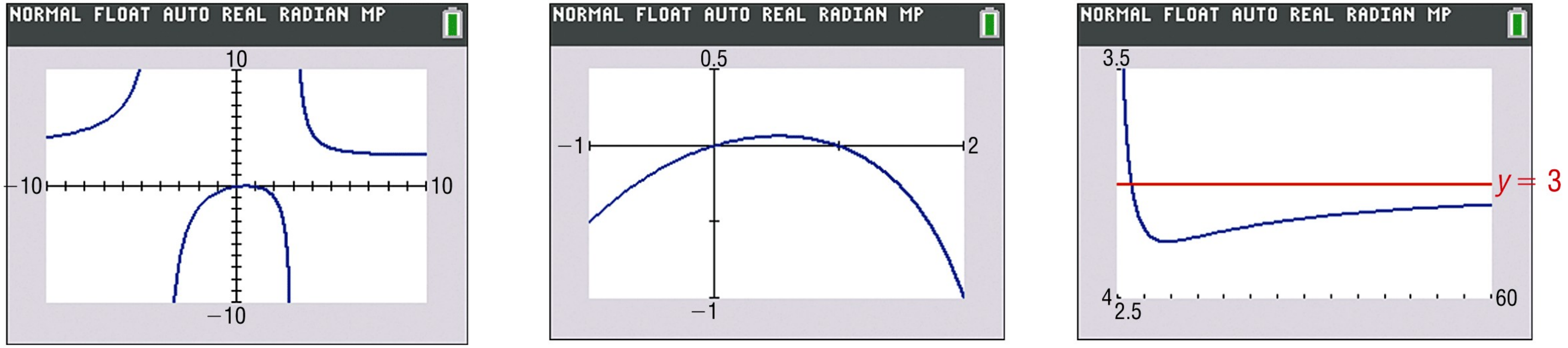
(b)



(c)

Figure

Graph: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$



(a) (b) (c)

Example

Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution

STEP 1: Factor R and obtain

$$R(x) = \frac{(2x-1)(x-2)}{(x+2)(x-2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

STEP 2: In lowest terms,

$$R(x) = \frac{2x-1}{x+2} \qquad x \neq -2, x \neq 2$$

Solution continued

STEP 3: The y-intercept is $R(0) = -\frac{1}{2}$. Plot the point $\left(0, -\frac{1}{2}\right)$. The graph has one x-intercept, $\frac{1}{2}$, with odd multiplicity. Plot the point $\left(\frac{1}{2}, 0\right)$. The graph will cross the x-axis at $x = \frac{1}{2}$. See Figure 36(a) on page 363.

STEP 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms*, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. Graph the line $x = -2$ using dashes. The multiplicity of -2 is odd, so the graph will approach ∞ on one side of the vertical asymptote and $-\infty$ on the other side.

STEP 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. Graph the line $y = 2$ using dashes.

Solution continued

To find out whether the graph of R intersects the horizontal asymptote $y = 2$, solve the equation $R(x) = 2$.

$$R(x) = \frac{2x - 1}{x + 2} = 2$$

$$2x - 1 = 2(x + 2)$$

$$2x - 1 = 2x + 4$$

$$-1 = 4 \qquad \text{Impossible}$$

The graph does not intersect the line $y = 2$.

STEP 6: Look at the factored expression for R in Step 1. The real zeros of the numerator and denominator, $-2, \frac{1}{2}$, and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \qquad \left(-2, \frac{1}{2}\right) \qquad \left(\frac{1}{2}, 2\right) \qquad (2, \infty)$$

Construct Table 13. Plot the points in Table 13.

Solution continued:Table 13

$-2 \qquad \frac{1}{2} \qquad 2 \qquad x$

Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of *R*	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above *x*-axis	Below *x*-axis	Above *x*-axis	Above *x*-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

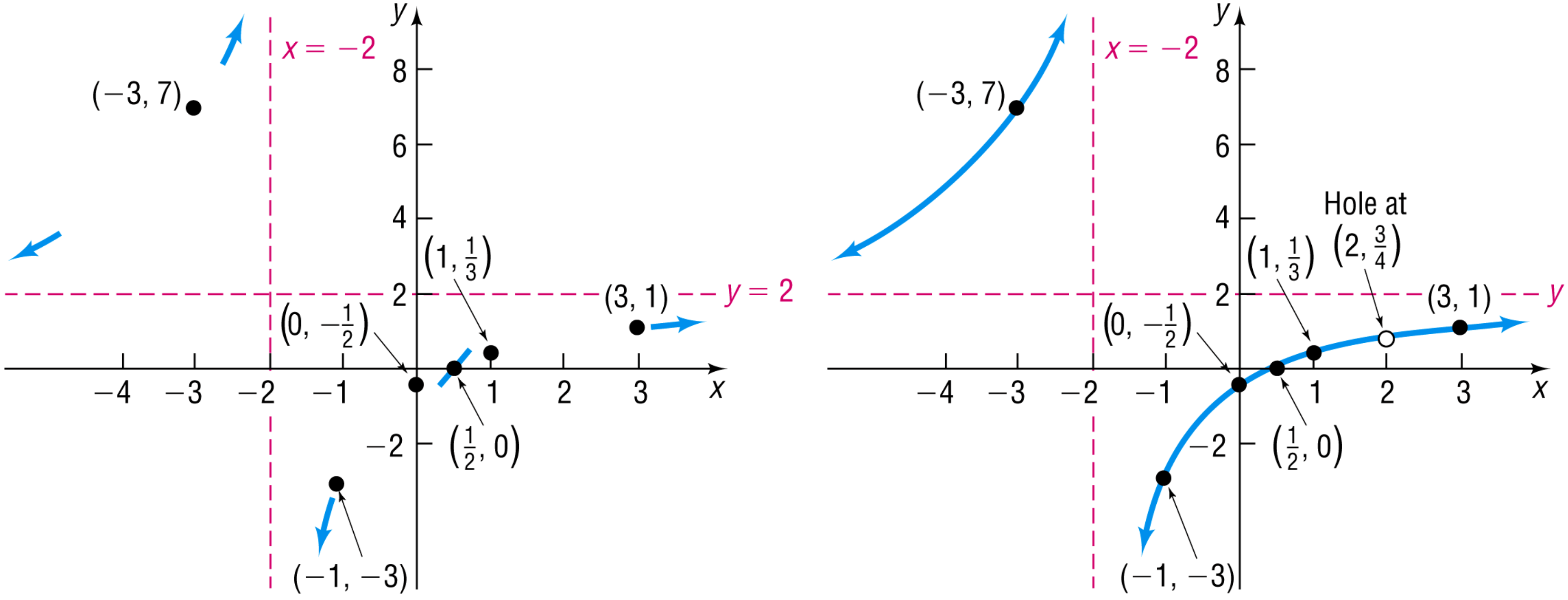
Solution continued

Step 7: From Table 13 we know that the graph of R is above the x-axis for $x < -2$. From Step 5 we know that the graph of R does not intersect the asymptote $y = 2$. Therefore, the graph of R will approach $y = 2$ from above as $x \to -\infty$ and will approach the vertical asymptote $x = -2$ at the top from the left.

Since the graph of R is below the x-axis for $-2 < x < \frac{1}{2}$, the graph will approach $x = -2$ at the bottom from the right. Finally, since the graph of R is above the x-axis for $x > \frac{1}{2}$ and does not intersect the horizontal asymptote $y = 2$, the graph of R will approach $y = 2$ from below as $x \to \infty$. The graph crosses the x-axis at $x = \frac{1}{2}$, changing from being below the x-axis to being above. See Figure 36(a).

See Figure 36(b) for the complete graph. Since R is not defined at 2, there is a hole at the point $\left(2, \frac{3}{4}\right)$.

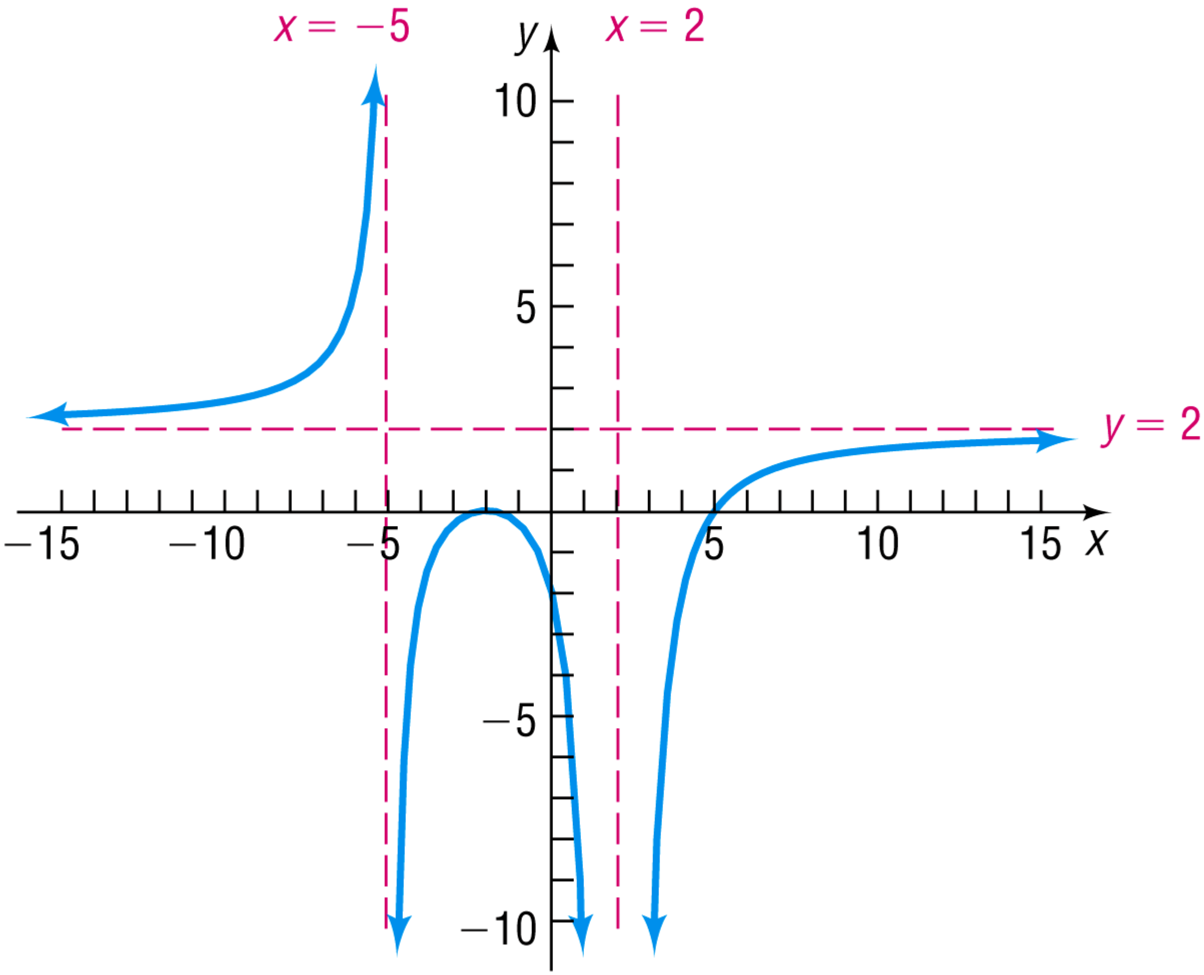
Solution continued: Figure 36



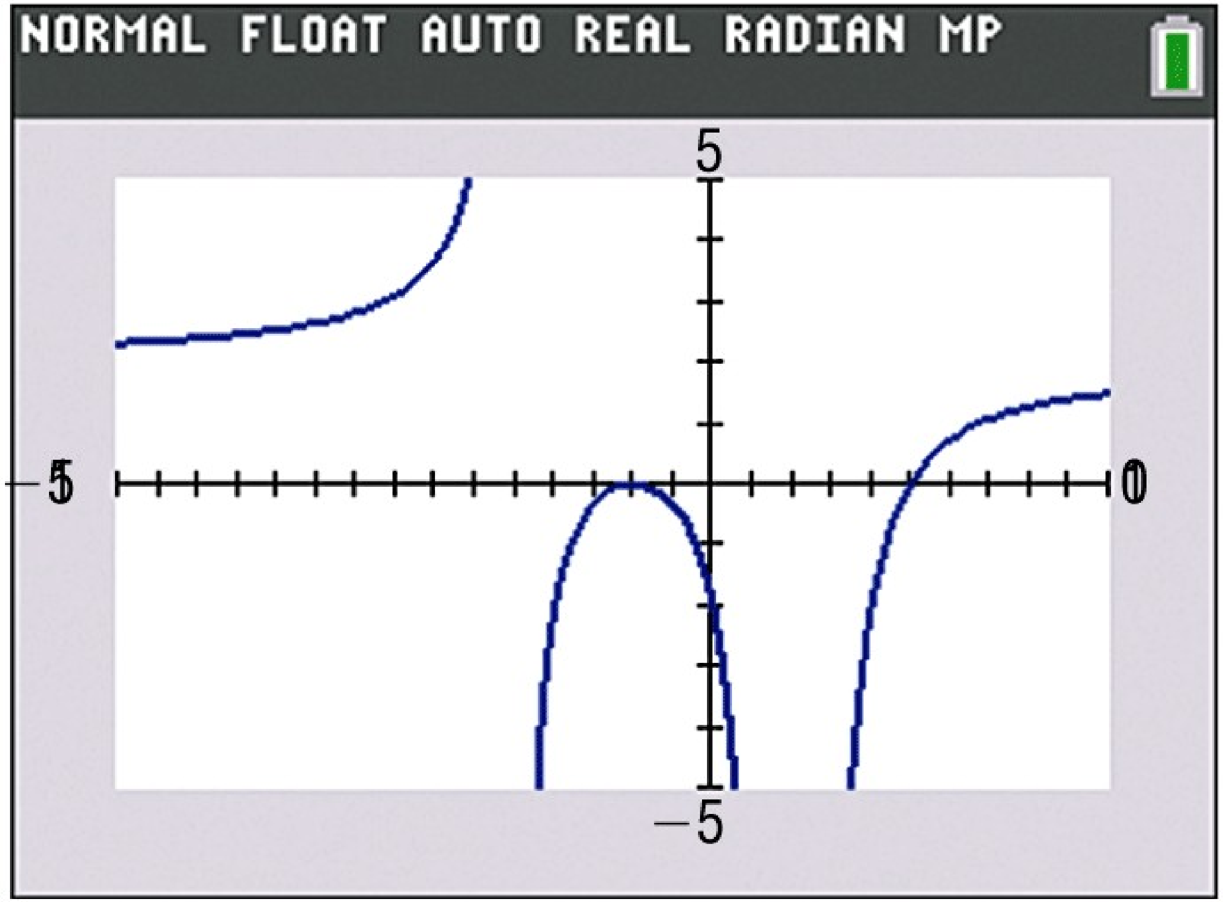
(a)

(b)

Figure



Figure



Solve Applied Problems Involving Rational Functions

Example

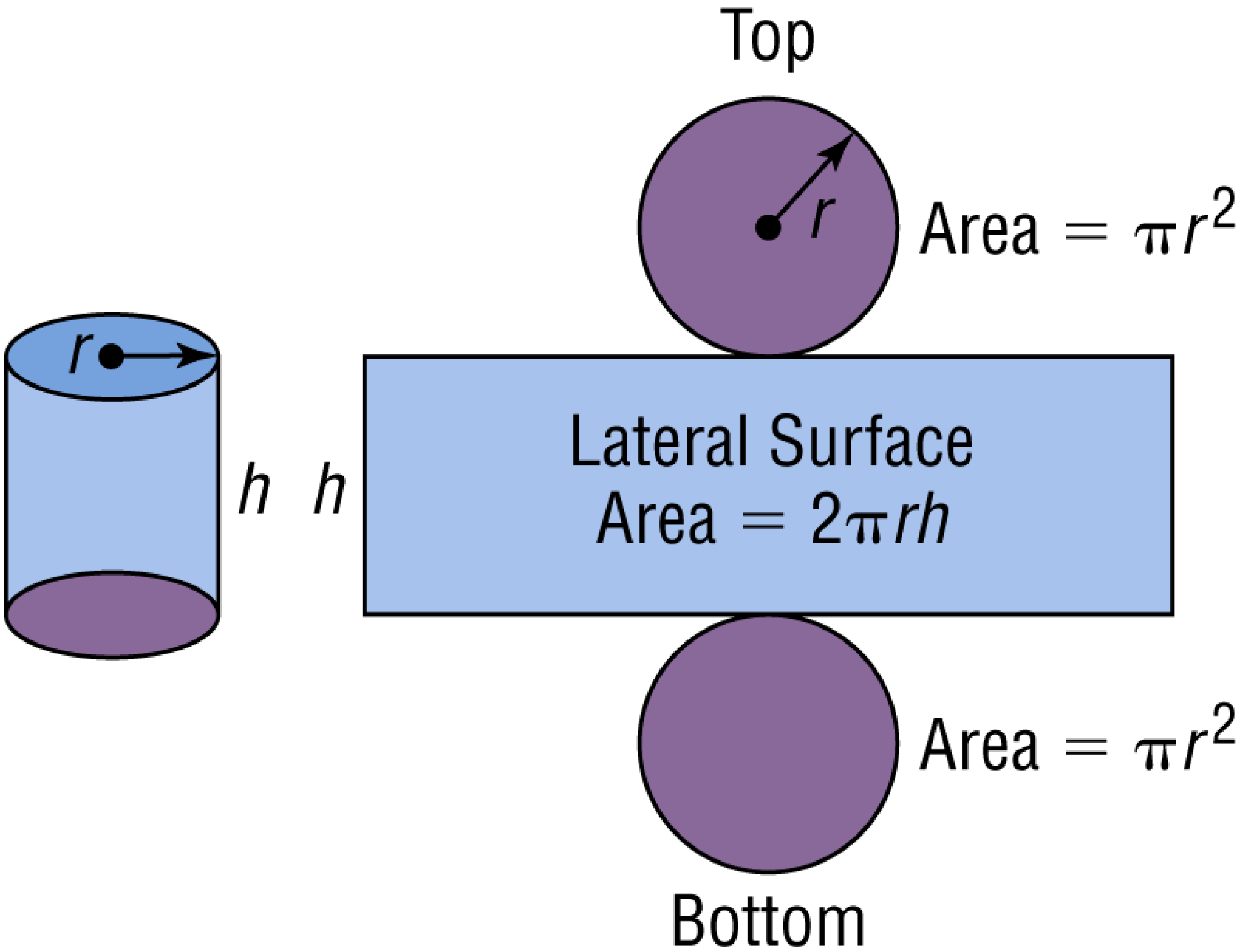
Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

Solution

Figure 39



Solution continued

(a) Figure 39 illustrates the components of a can in the shape of a right circular cylinder. Notice that the material required to produce a cylindrical can of height h and radius r consists of a rectangle of area $2\pi rh$ and two circles, each of area πr^2. The total cost C (in cents) of manufacturing the can is therefore

$$C = \text{Cost of the top and bottom} + \text{Cost of the side}$$

$$= \underbrace{2(\pi r^2)}_{\substack{\text{Total area}\\ \text{of top and}\\ \text{bottom}}} \cdot \underbrace{(0.05)}_{\substack{\text{Cost/unit}\\ \text{area}}} + \underbrace{(2\pi rh)}_{\substack{\text{Total}\\ \text{area of}\\ \text{side}}} \cdot \underbrace{(0.02)}_{\substack{\text{Cost/unit}\\ \text{area}}}$$

$$= 0.10\pi r^2 + 0.04\pi rh$$

There is an additional restriction that the height h and radius r must be chosen so that the volume V of the can is 500 cubic centimeters. Since $V = \pi r^2 h$, we have

$$500 = \pi r^2 h \quad \text{so} \quad h = \frac{500}{\pi r^2}$$

Substituting this expression for h, we find that the cost C, in cents, as a function of the radius r is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \cdot \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

(b) See Figure 40 for the graph of $C = C(r)$.

(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 centimeters.

(d) The least cost is $C(3.17) \approx 9.47¢$.

Solution continued: Figure 40