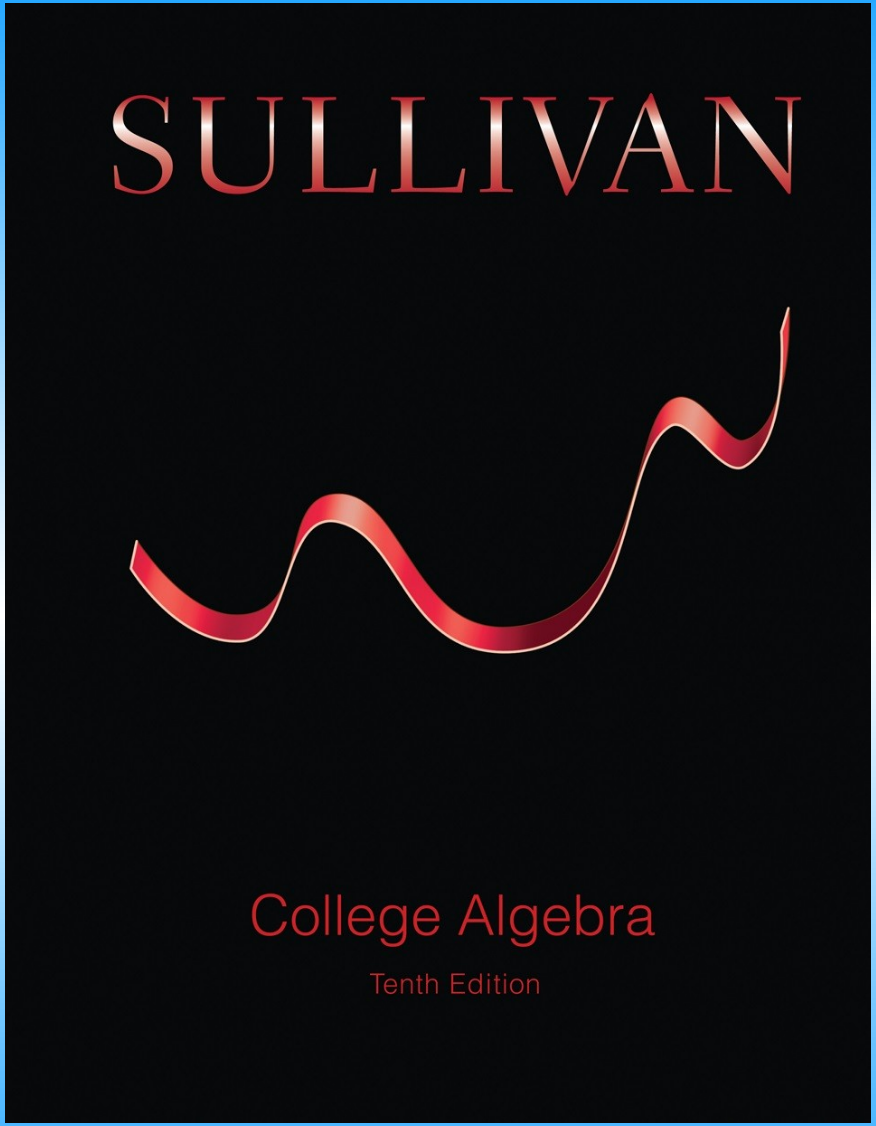
Chapter 4

Section 1



4.1 Properties of Linear Functions and Linear Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Lines (Section 2.3, pp. 167–175)
- Graphs of Equations in Two Variables; Intercepts; Symmetry (Section 2.2, pp. 157–164)
- Linear Equations (Section 1.1, pp. 82–87)
- Functions (Section 3.1, pp. 199–208)
- The Graph of a Function (Section 3.2, pp. 214–217)
- Properties of Functions (Section 3.3, pp. 223–231)

Now Work the **'Are You Prepared?'** problems on page 280.

OBJECTIVES

1. Graph Linear Functions (p. 274)
2. Use Average Rate of Change to Identify Linear Functions (p. 274)
3. Determine Whether a Linear Function Is Increasing, Decreasing, or Constant (p. 277)
4. Build Linear Models from Verbal Descriptions (p. 278)

Graph Linear Functions

Definition

A **linear function** is a function of the form

$$f(x) = mx + b$$

The graph of a linear function is a line with slope m and y-intercept b. Its domain is the set of all real numbers.

Example

Graphing a Linear Function

Graph the linear function $f(x) = -3x + 7$. What are the domain and the range of f?

Solution

This is a linear function with slope $m = -3$ and y-intercept $b = 7$. To graph this function, plot the point $(0, 7)$, the y-intercept, and use the slope to find an additional point by moving right 1 unit and down 3 units. See Figure 1. The domain and the range of f are each the set of all real numbers.

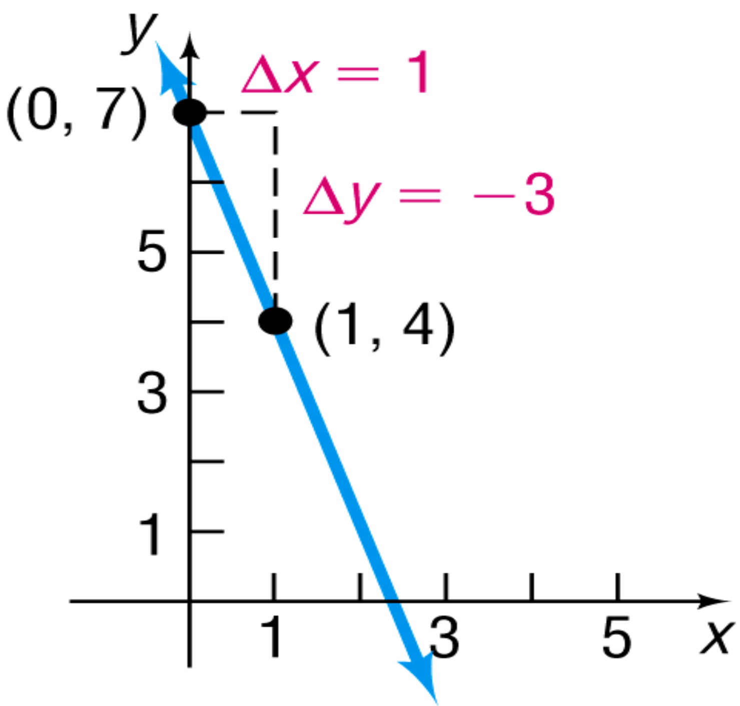


Figure 1

Use Average Rate of Change to Identify Linear Functions

Table

x	$y = f(x) = -3x + 7$	Average Rate of Change $= \dfrac{\Delta y}{\Delta x}$
-2	13	
		$\dfrac{10 - 13}{-1 - (-2)} = \dfrac{-3}{1} = -3$
-1	10	
		$\dfrac{7 - 10}{0 - (-1)} = \dfrac{-3}{1} = -3$
0	7	
		-3
1	4	
		-3
2	1	
		-3
3	-2	

Theorem

Average Rate of Change of a Linear Function

Linear functions have a constant average rate of change. That is, the average rate of change of a linear function $f(x) = mx + b$ is

$$\frac{\Delta y}{\Delta x} = m$$

Example

Using the Average Rate of Change to Identify Linear Functions

(a) A strain of *E. coli* known as Beu 397-recA441 is placed into a Petri dish at 30° Celsius and allowed to grow. The data shown in Table 2 on the next page are collected. The population is measured in grams and the time in hours. Plot the ordered pairs (x, y) in the Cartesian plane, and use the average rate of change to determine whether the function is linear.

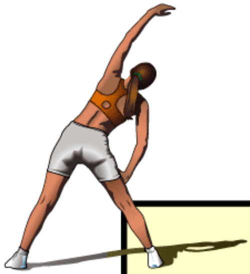
(b) The data in Table 3 represent the maximum number of heartbeats that a healthy individual of different ages should have during a 15-second interval of time while exercising. Plot the ordered pairs (x, y) in the Cartesian plane, and use the average rate of change to determine whether the function is linear.

Example continued

Time (hours), x	Population (grams), y	(x, y)
0	0.09	(0, 0.09)
1	0.12	(1, 0.12)
2	0.16	(2, 0.16)
3	0.22	(3, 0.22)
4	0.29	(4, 0.29)
5	0.39	(5, 0.39)

Table 2

Example continued



Age, *x*	Maximum Number of Heartbeats, *y*	(*x*, *y*)
20	50	(20, 50)
30	47.5	(30, 47.5)
40	45	(40, 45)
50	42.5	(50, 42.5)
60	40	(60, 40)
70	37.5	(70, 37.5)

Table 3

Source: American Heart Association

Solution

Compute the average rate of change of each function. If the average rate of change is constant, the function is linear. If the average rate of change is not constant, the function is nonlinear.

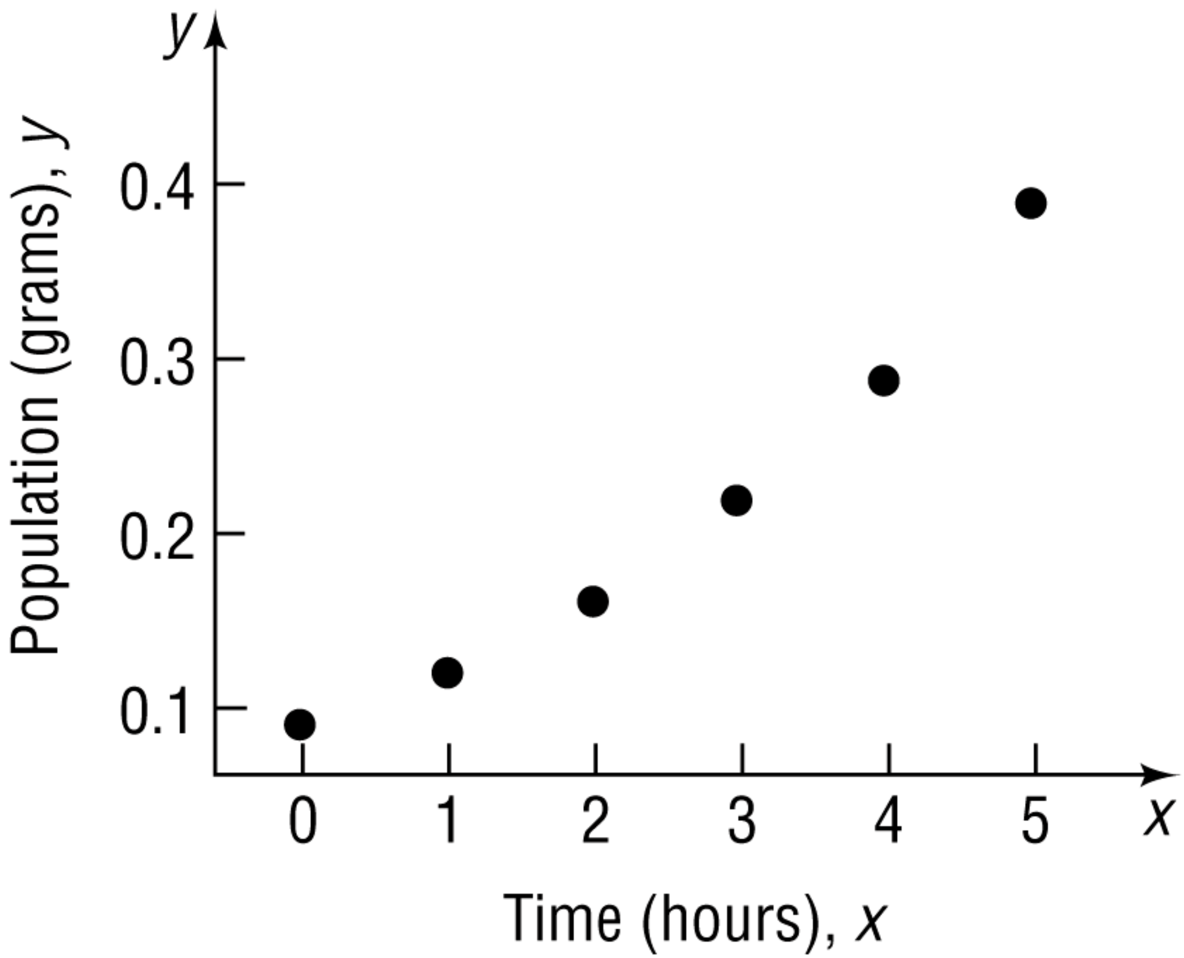
(a) Figure 2 shows the points listed in Table 2 plotted in the Cartesian plane. Note that it is impossible to draw a straight line that contains all the points. Table 4 displays the average rate of change of the population.

Because the average rate of change is not constant, the function is not linear. In fact, because the average rate of change is increasing as the value of the independent variable increases, the function is increasing at an increasing rate. So not only is the population increasing over time, but it is also growing more rapidly as time passes.

(b) Figure 3 shows the points listed in Table 3 plotted in the Cartesian plane. Note that the data in Figure 3 lie on a straight line. Table 5 displays the average rate of change of the maximum number of heartbeats. The average rate of change of the heartbeat data is constant, -0.25 beat per year, so the function is linear.

Solution continued

Part a:
Figure 2



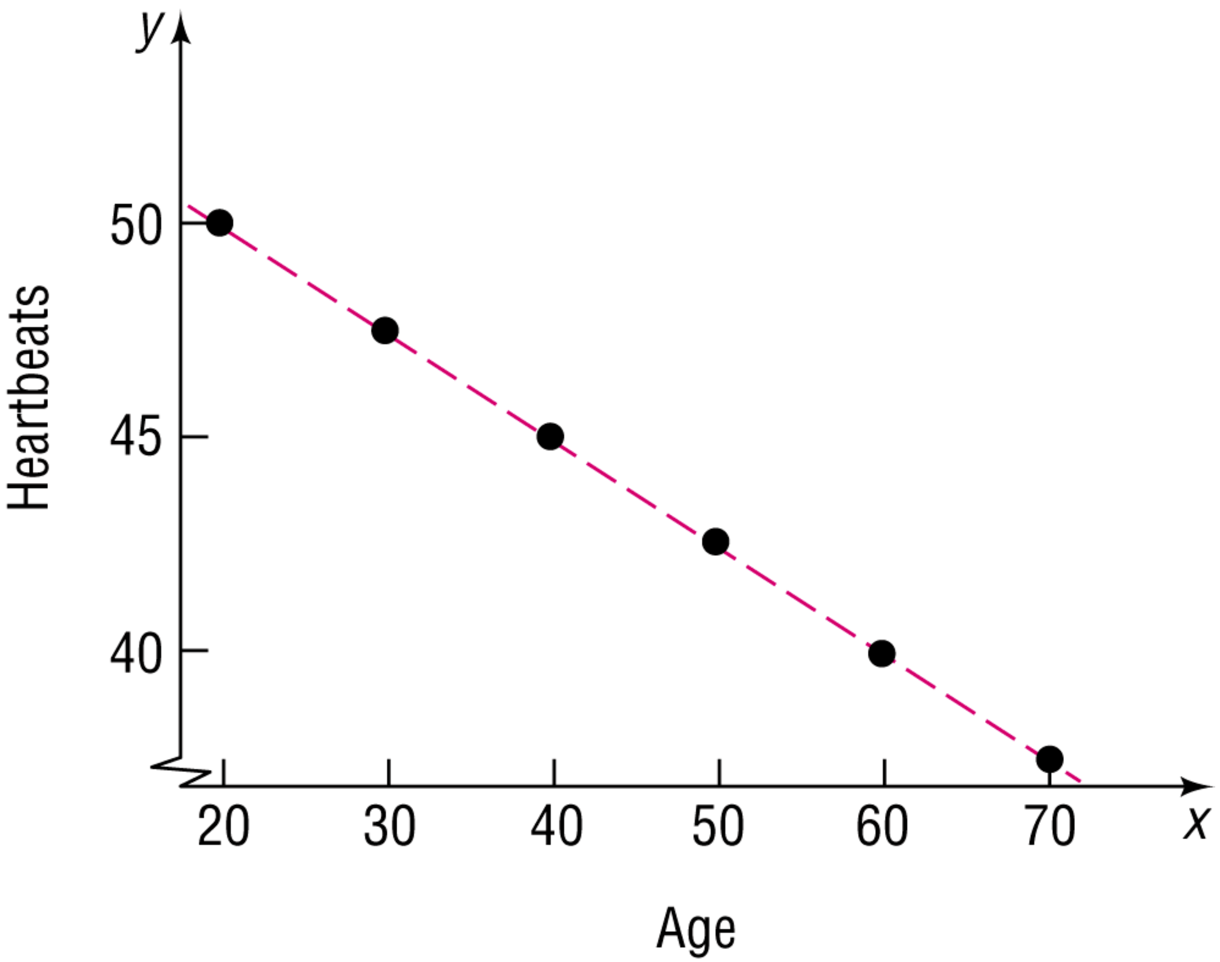
Solution continued

Part a:
Table 4

Time (hours), x	Population (grams), y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
0	0.09	
		$\frac{0.12 - 0.09}{1 - 0} = 0.03$
1	0.12	
		0.04
2	0.16	
		0.06
3	0.22	
		0.07
4	0.29	
		0.10
5	0.39	

Solution continued

Part b:
Figure 3



Solution continued

Part b:
Table 5

Age, x	Maximum Number of Heartbeats, y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
20	50	
		$\frac{47.5 - 50}{30 - 20} = -0.25$
30	47.5	
		-0.25
40	45	
		-0.25
50	42.5	
		-0.25
60	40	
		-0.25
70	37.5	

Determine Whether a Linear Function Is Increasing, Decreasing, or Constant

Theorem

Increasing, Decreasing, and Constant Linear Functions

A linear function $f(x) = mx + b$ is increasing over its domain if its slope, m, is positive. It is decreasing over its domain if its slope, m, is negative. It is constant over its domain if its slope, m, is zero.

Example

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

Determine whether the following linear functions are increasing, decreasing, or constant.

(a) $f(x) = 5x - 2$

(b) $g(x) = -2x + 8$

(c) $s(t) = \frac{3}{4}t - 4$

(d) $h(z) = 7$

Solution

(a) For the linear function $f(x) = 5x - 2$, the slope is 5, which is positive. The function f is increasing on the interval $(-\infty, \infty)$.

(b) For the linear function $g(x) = -2x + 8$, the slope is -2, which is negative. The function g is decreasing on the interval $(-\infty, \infty)$.

(c) For the linear function $s(t) = \frac{3}{4}t - 4$, the slope is $\frac{3}{4}$, which is positive. The function s is increasing on the interval $(-\infty, \infty)$.

(d) The linear function h can be written as $h(z) = 0z + 7$. Because the slope is 0, the function h is constant on the interval $(-\infty, \infty)$.

Build Linear Models from Verbal Descriptions

Modeling with a Linear Function

If the average rate of change of a function is a constant m, a linear function f can be used to model the relation between the two variables as follows:

$$f(x) = mx + b$$

where b is the value of f at 0; that is, $b = f(0)$.

Example

Straight-line Depreciation

Book value is the value of an asset that a company uses to create its balance sheet. Some companies depreciate assets using straight-line depreciation so that the value of the asset declines by a fixed amount each year. The amount of the decline depends on the useful life that the company assigns to the asset. Suppose a company just purchased a fleet of new cars for its sales force at a cost of \$31,500 per car. The company chooses to depreciate each vehicle using the straight-line method over 7 years. This means that each car will depreciate by $\frac{\$31,500}{7} = \$4500$ per year.

(a) Write a linear function that expresses the book value V of each car as a function of its age, x, in years.

(b) Graph the linear function.

(c) What is the book value of each car after 3 years?

(d) Interpret the slope.

(e) When will the book value of each car be \$9000? [**Hint:** Solve the equation $V(x) = 9000$.]

Solution

(a) If we let $V(x)$ represent the value of each car after x years, then $V(0)$ represents the original value of each car, so $V(0) = \$31{,}500$. The y-intercept of the linear function is \$31,500. Because each car depreciates by \$4500 per year, the slope of the linear function is -4500. The linear function that represents the book value V of each car after x years is

$$V(x) = -4500x + 31{,}500$$

(b) Figure 4 shows the graph of V.

(c) The book value of each car after 3 years is

$$\begin{aligned} V(3) &= -4500(3) + 31{,}500 \\ &= \$18{,}000 \end{aligned}$$

(d) Since the slope of $V(x) = -4500x + 31{,}500$ is -4500, the average rate of change of the book value is $-\$4500$/year. So for each additional year that passes, the book value of the car decreases by \$4500.

Solution continued

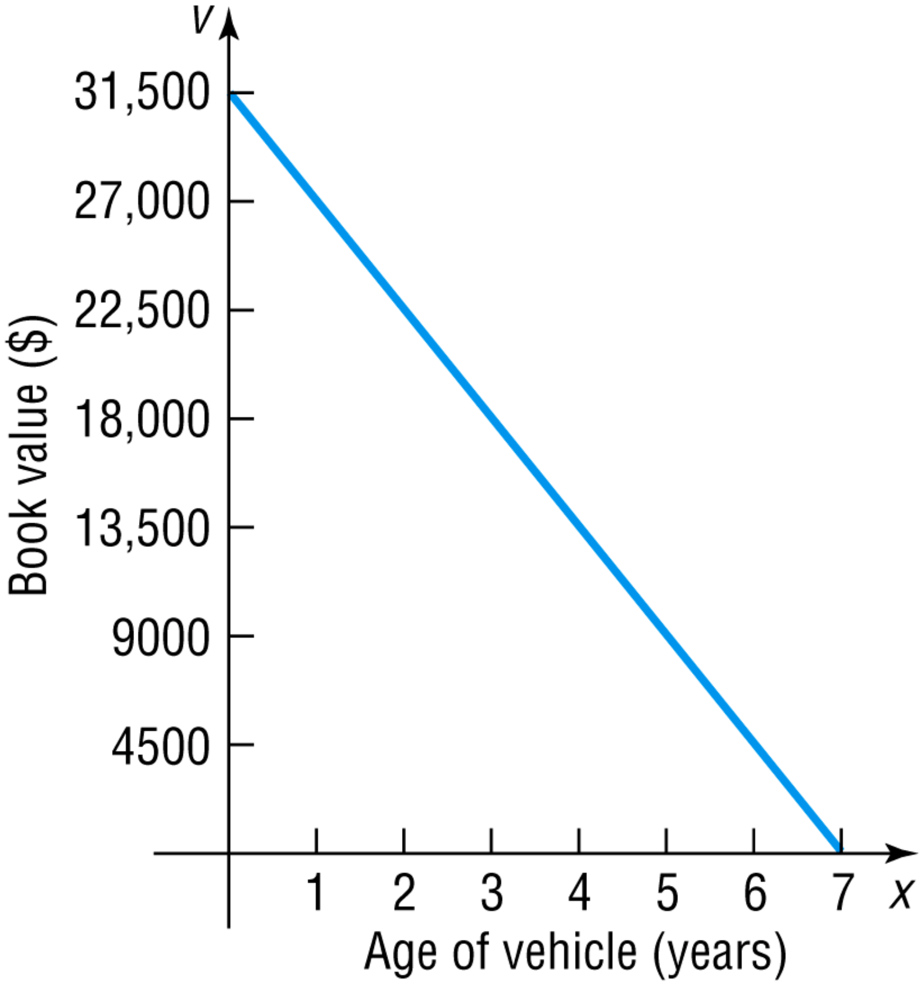
(e) To find when the book value will be $9000, solve the equation

$$\begin{aligned} V(x) &= 9000 \\ -4500x + 31{,}500 &= 9000 \\ -4500x &= -22{,}500 \quad \text{Subtract 31,500 from each side.} \\ x &= \frac{-22{,}500}{-4500} = 5 \quad \text{Divide by } -4500. \end{aligned}$$

The car will have a book value of $9000 when it is 5 years old.

Solution continued

Figure 4



Example

Supply and Demand

The **quantity supplied** of a good is the amount of a product that a company is willing to make available for sale at a given price. The **quantity demanded** of a good is the amount of a product that consumers are willing to purchase at a given price. Suppose that the quantity supplied, S, and the quantity demanded, D, of widgets are given by the following functions:

$$S(p) = 40p - 200$$
$$D(p) = -20p + 1000$$

where p is the price (in dollars) of a widget.

Example continued

(a) The **equilibrium price** of a product is defined as the price at which quantity supplied equals quantity demanded. That is, the equilibrium price is the price at which $S(p) = D(p)$. Find the equilibrium price of widgets.

What is the **equilibrium quantity**, the amount demanded (or supplied) at the equilibrium price?

Example continued

(b) Determine the prices for which quantity supplied is greater than quantity demanded. That is, solve the inequality $S(p) > D(p)$.

(c) Graph $S = S(p)$ and $D = D(p)$, and label the **equilibrium point**, the point of intersection of S and D.

Solution

(a) To find the equilibrium price, solve the equation $S(p) = D(p)$.

$40p - 200 = -20p + 1000$	$S(p) = 40p - 200$ $D(p) = -20p + 1000$
$40p = -20p + 1200$	Add 200 to each side.
$60p = 1200$	Add $20p$ to each side.
$p = 20$	Divide each side by 60.

The equilibrium price is $20 per widget.

Solution continued

To find the equilibrium quantity, evaluate either $S(p)$ or $D(p)$ at $p = 20.$

$$S(20) = 40(20) - 200 = 600$$

The equilibrium quantity is 600 widgets. At a price of $20 per widget, the company will produce and sell 600 widgets each month and have no shortages or excess inventory.

Solution continued

(b) The inequality $S(p) > D(p)$ is

$$\begin{aligned} 40p - 200 &> -20p + 1000 && S(p) > D(p) \\ 40p &> -20p + 1200 && \text{Add 200 to each side.} \\ 60p &> 1200 && \text{Add } 20p \text{ to each side.} \\ p &> 20 && \text{Divide each side by 60.} \end{aligned}$$

If the company charges more than \$20 per widget, quantity supplied will exceed quantity demanded. In this case the company will have excess widgets in inventory.

Solution continued

(c) The figure shows the graphs of $S = S(p)$ and $D = D(p)$ with the equilibrium point labeled.