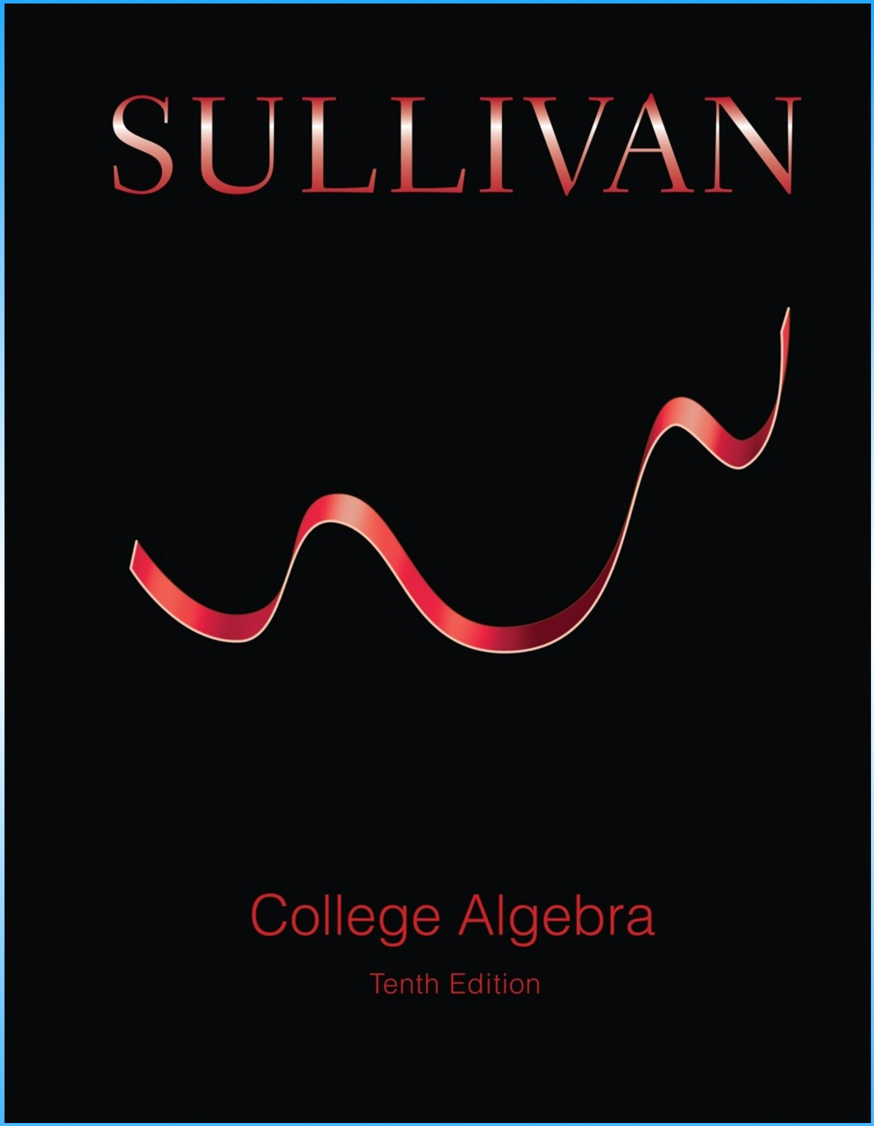
Chapter 3

Section 3



3.3 Properties of Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Section 1.5, pp. 120–121)
- Intercepts (Section 2.2, pp. 159–160)
- Slope of a Line (Section 2.3, pp. 167–169)
- Point–Slope Form of a Line (Section 2.3, p. 171)
- Symmetry (Section 2.2, pp. 160–162)

Now Work the **'Are You Prepared?'** problems on page 232.

OBJECTIVES

1. Determine Even and Odd Functions from a Graph (p. 223)
2. Identify Even and Odd Functions from an Equation (p. 225)
3. Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant (p. 225)
4. Use a Graph to Locate Local Maxima and Local Minima (p. 226)
5. Use a Graph to Locate the Absolute Maximum and the Absolute Minimum (p. 227)
6. Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing (p. 229)
7. Find the Average Rate of Change of a Function (p. 230)

Determine Even and Odd Functions from a Graph

Definition

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = f(x)}$$

Definition

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = -f(x)}$$

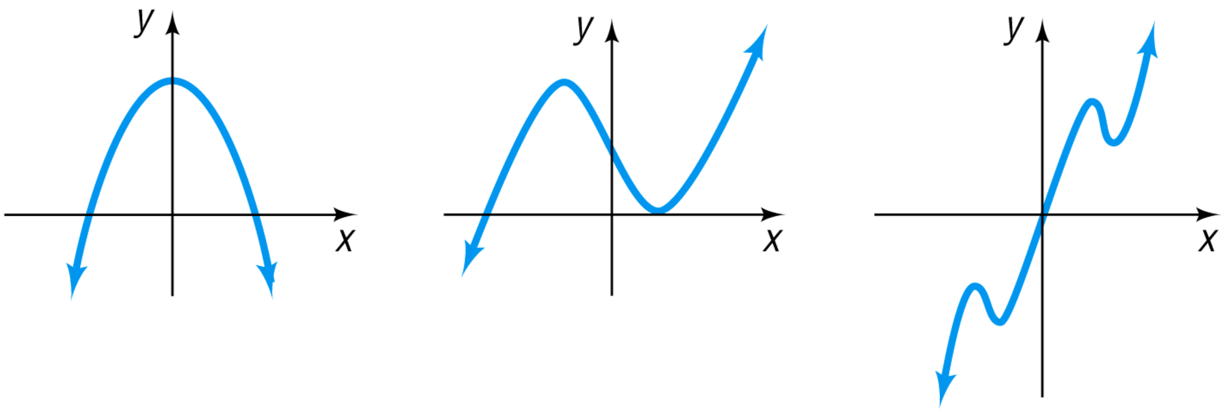
Theorem

A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

Example

Determining Even and Odd Functions from the Graph

Determine whether each graph given in Figure 17 is the graph of an even function, an odd function, or a function that is neither even nor odd.



(a) (b) (c)

Figure 17

Solution

(a) The graph in Figure 17(a) is that of an even function, because the graph is symmetric with respect to the y-axis.

(b) The function whose graph is given in Figure 17(b) is neither even nor odd, because the graph is neither symmetric with respect to the y-axis nor symmetric with respect to the origin.

(c) The function whose graph is given in Figure 17(c) is odd, because its graph is symmetric with respect to the origin.

Identify Even and Odd Functions from an Equation

Example

Identifying Even and Odd Functions Algebraically

Determine whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the y-axis, with respect to the origin, or neither.

(a) $f(x) = x^2 - 5$

(b) $g(x) = x^3 - 1$

(c) $h(x) = 5x^3 - x$

(d) $F(x) = |x|$

Solution

(a) To determine whether f is even, odd, or neither, replace x by $-x$ in $f(x) = x^2 - 5$.

$$f(-x) = (-x)^2 - 5 = x^2 - 5 = f(x)$$

Since $f(-x) = f(x)$, the function is even, and the graph of f is symmetric with respect to the y-axis.

(b) Replace x by $-x$ in $g(x) = x^3 - 1$.

$$g(-x) = (-x)^3 - 1 = -x^3 - 1$$

Since $g(-x) \neq g(x)$ and $g(-x) \neq -g(x) = -(x^3 - 1) = -x^3 + 1$, the function is neither even nor odd. The graph of g is not symmetric with respect to the y-axis, nor is it symmetric with respect to the origin.

Solution continued

(c) Replace x by $-x$ in $h(x) = 5x^3 - x$.

$$h(-x) = 5(-x)^3 - (-x) = -5x^3 + x = -(5x^3 - x) = -h(x)$$

Since $h(-x) = -h(x)$, h is an odd function, and the graph of h is symmetric with respect to the origin.

(d) Replace x by $-x$ in $F(x) = |x|$.

$$F(-x) = |-x| = |-1| \cdot |x| = |x| = F(x)$$

Since $F(-x) = F(x)$, F is an even function, and the graph of F is symmetric with respect to the y-axis.

Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

Example

Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Determine the values of x for which the function in Figure 18 is increasing. Where is it decreasing? Where is it constant?

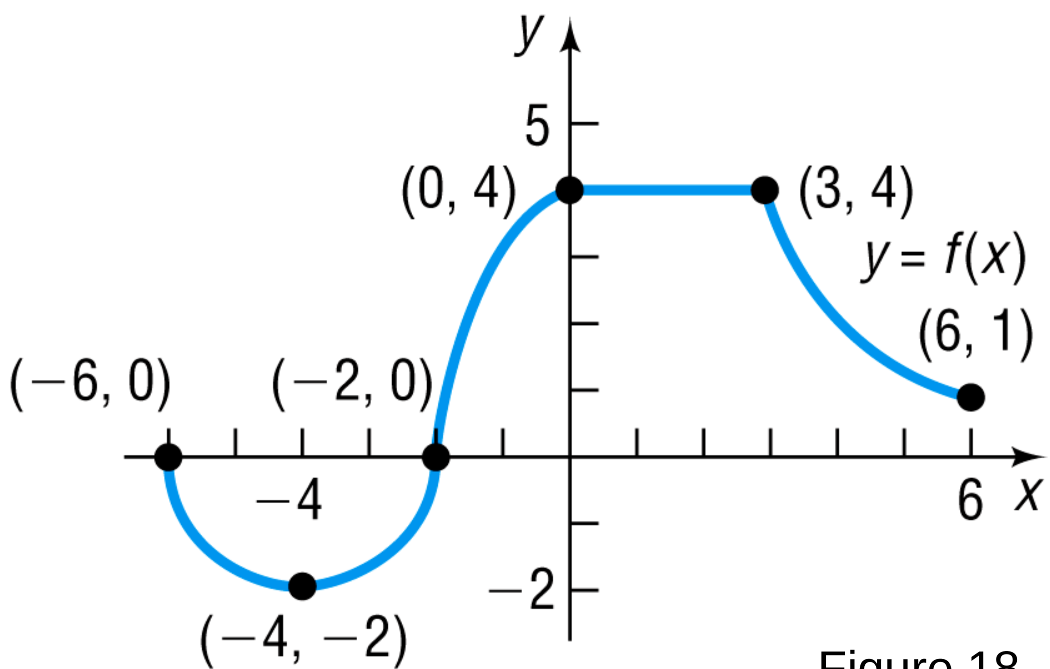


Figure 18

Solution

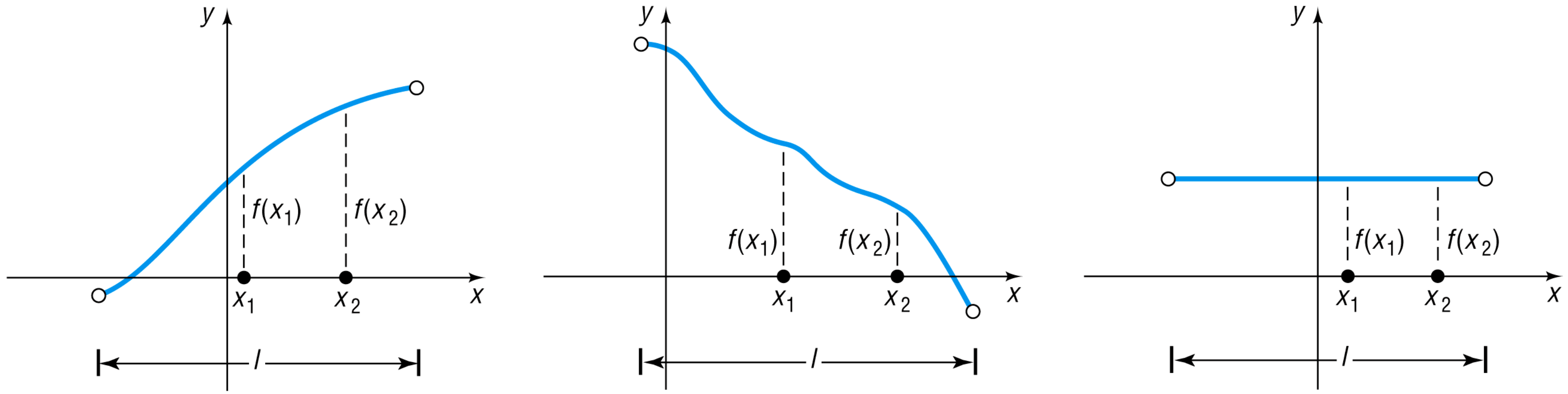
When determining where a function is increasing, where it is decreasing, and where it is constant, we use strict inequalities involving the independent variable x, or we use open intervals* of x-coordinates. The function whose graph is given in Figure 18 is increasing on the open interval $(-4, 0)$, or for $-4 < x < 0$. The function is decreasing on the open intervals $(-6, -4)$ and $(3, 6)$, or for $-6 < x < -4$ and $3 < x < 6$. The function is constant on the open interval $(0, 3)$, or for $0 < x < 3$.

DEFINITIONS A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.

Figure: Illustration of the Definitions



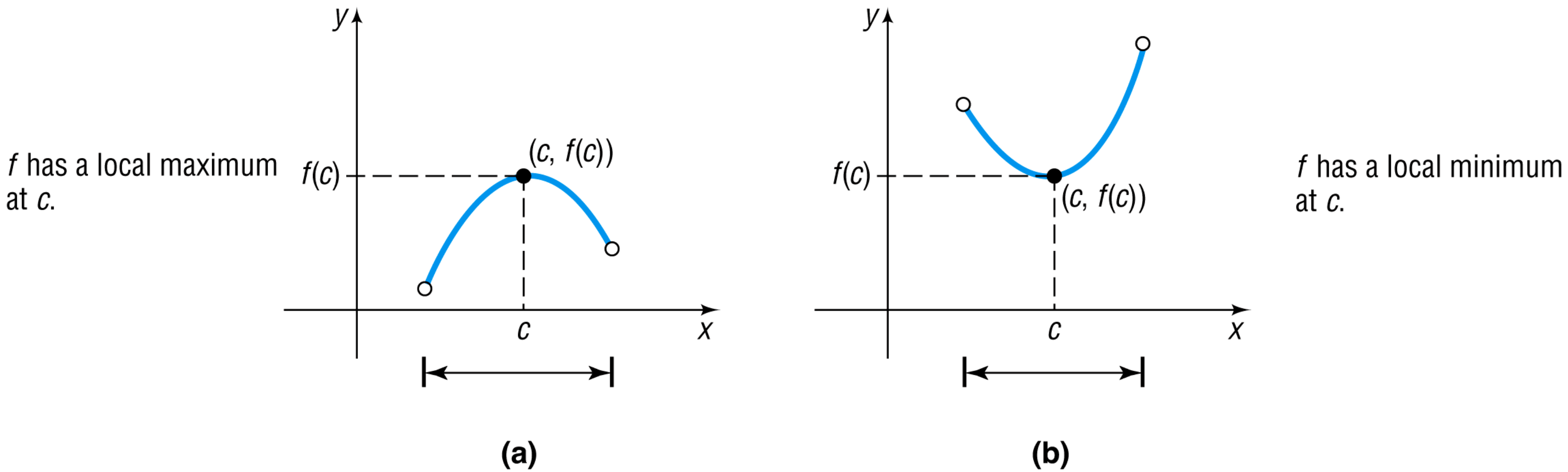
(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I.

(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I.

(c) For all x in I, the values of f are equal; f is constant on I.

Use a Graph to Locate Local Maxima and Local Minima

Figure: Local maximum and local minimum



Definition

Let f be a function defined on some interval I.

A function f has a **local maximum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \leq f(c)$. We call $f(c)$ a **local maximum value of f**.

A function f has a **local minimum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \geq f(c)$. We call $f(c)$ a **local minimum value of f**.

Example

Finding Local Maxima and Local Minima from the Graph of a Function and Determining Where the Function Is Increasing, Decreasing, or Constant

Figure 21 shows the graph of a function f.

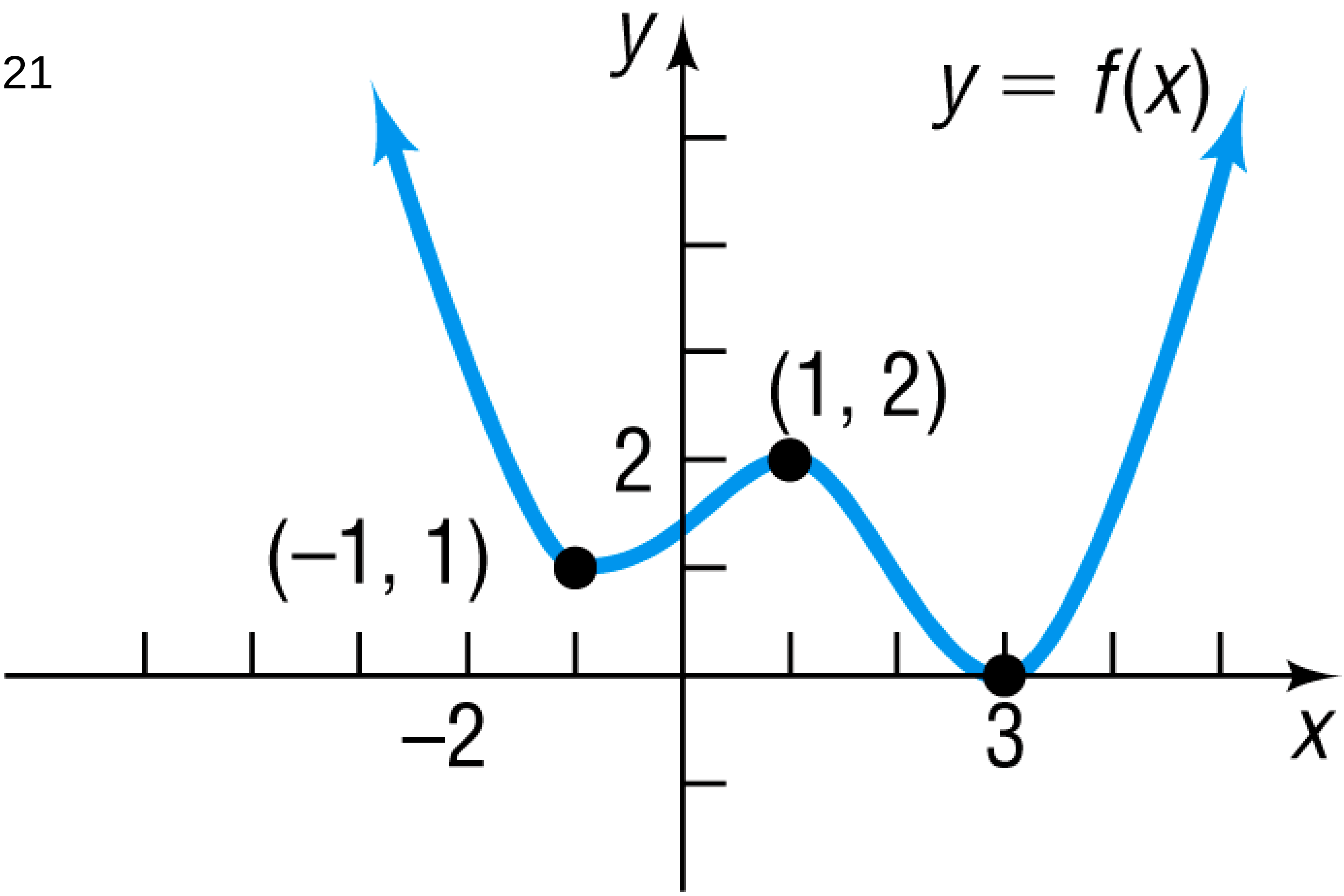
(a) At what value(s) of x, if any, does f have a local maximum? List the local maximum values.

(b) At what value(s) of x, if any, does f have a local minimum? List the local minimum values.

(c) Find the intervals on which f is increasing. Find the intervals on which f is decreasing.

Example continued

Figure 21



Solution

The domain of f is the set of real numbers.

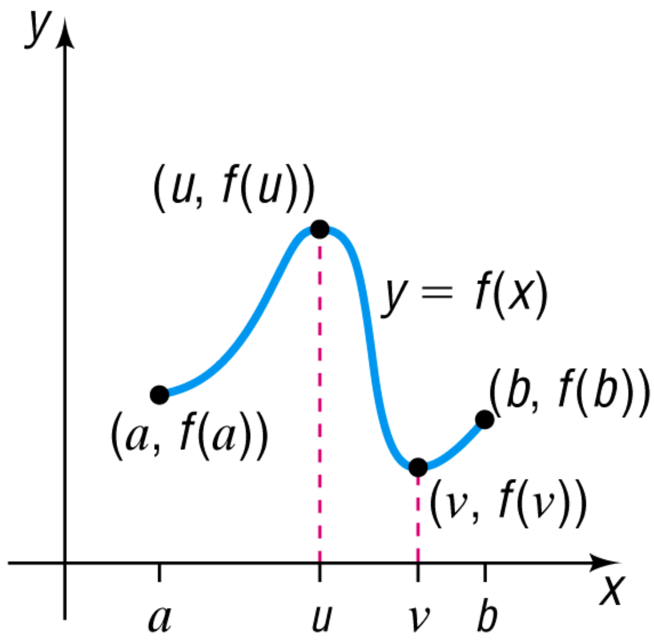
(a) f has a local maximum at 1, since for all x close to 1, we have $f(x) \leq f(1)$. The local maximum value is $f(1) = 2$.

(b) f has local minima at -1 and at 3. The local minimum values are $f(-1) = 1$ and $f(3) = 0$.

(c) The function whose graph is given in Figure 21 is increasing for all values of x between -1 and 1 and for all values of x greater than 3. That is, the function is increasing on the intervals $(-1, 1)$ and $(3, \infty)$, or for $-1 < x < 1$ and $x > 3$. The function is decreasing for all values of x less than -1 and for all values of x between 1 and 3. That is, the function is decreasing on the intervals $(-\infty, -1)$ and $(1, 3)$, or for $x < -1$ and $1 < x < 3$.

Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

Figure



domain: $[a, b]$
for all x in $[a, b]$, $f(x) \leq f(u)$
for all x in $[a, b]$, $f(x) \geq f(v)$
absolute maximum: $f(u)$
absolute minimum: $f(v)$

DEFINITION Let f be a function defined on some interval I. If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then f has an **absolute maximum at** $\boldsymbol{u}$, and the number $f(u)$ is the **absolute maximum of** $\boldsymbol{f}$ **on** $\boldsymbol{I}$.

If there is a number v in I for which $f(x) \geq f(v)$ for all x in I, then f has an **absolute minimum at** $\boldsymbol{v}$, and the number $f(v)$ is the **absolute minimum of** $\boldsymbol{f}$ **on** $\boldsymbol{I}$.

Example

Finding the Absolute Maximum and the Absolute Minimum from the Graph of a Function

For each graph of a function $y = f(x)$ in Figure 23, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.

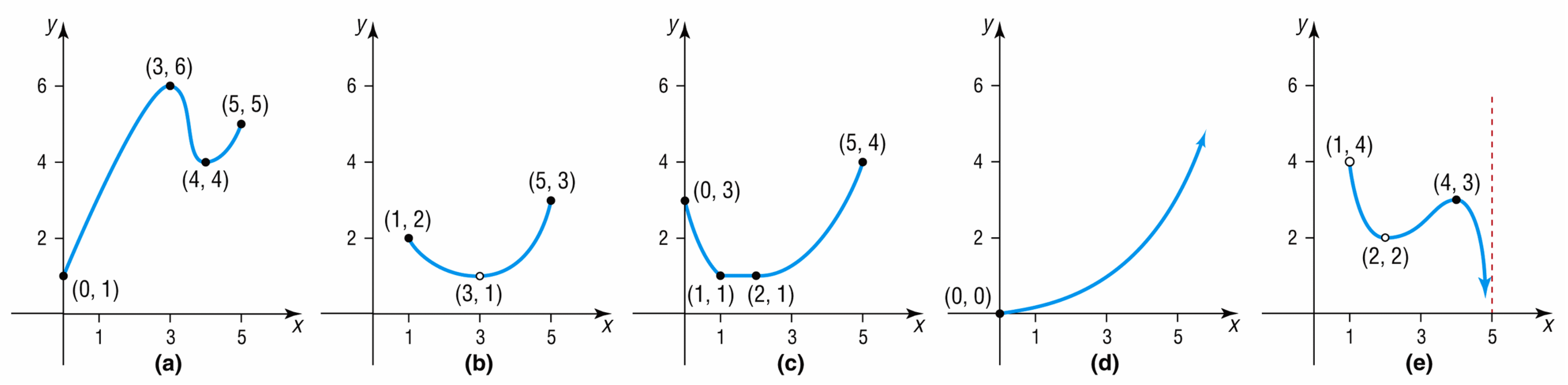


Figure 23

Solution

(a) The function f whose graph is given in Figure 23(a) has the closed interval $[0, 5]$ as its domain. The largest value of f is $f(3) = 6$, the absolute maximum. The smallest value of f is $f(0) = 1$, the absolute minimum. The function has a local maximum of 6 at $x = 3$ and a local minimum of 4 at $x = 4$.

(b) The function f whose graph is given in Figure 23(b) has the domain $\{x|1 \le x \le 5, x \ne 3\}$. Note that we exclude 3 from the domain because of the "hole" at (3, 1). The largest value of f on its domain is $f(5) = 3$, the absolute maximum. There is no absolute minimum. Do you see why? As you trace the graph, getting closer to the point (3, 1), there is no single smallest value. [As soon as you claim a smallest value, we can trace closer to (3, 1) and get a smaller value!] The function has no local maxima or minima.

Solution continued

(c) The function f whose graph is given in Figure 23(c) has the interval $[0, 5]$ as its domain. The absolute maximum of f is $f(5) = 4$. The absolute minimum is 1. Notice that the absolute minimum 1 occurs at any number in the interval $[1, 2]$. The function has a local minimum value of 1 at every x in the interval $[1, 2]$, but it has no local maximum value.

(d) The function f given in Figure 23(d) has the interval $[0, \infty)$ as its domain. The function has no absolute maximum; the absolute minimum is $f(0) = 0$. The function has no local maximum or local minimum.

(e) The function f in Figure 23(e) has the domain $\{x | 1 < x < 5, x \neq 2\}$. The function has no absolute maximum and no absolute minimum. Do you see why? The function has a local maximum value of 3 at $x = 4$, but no local minimum value.

Theorem

Extreme Value Theorem

If f is a continuous function* whose domain is a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing

Example

Using a Graphing Utility to Approximate Local Maxima and Minima and to Determine Where a Function Is Increasing or Decreasing

(a) Use a graphing utility to graph $f(x) = 6x^3 - 12x + 5$ for $-2 < x < 2$. Approximate where f has a local maximum and where f has a local minimum.

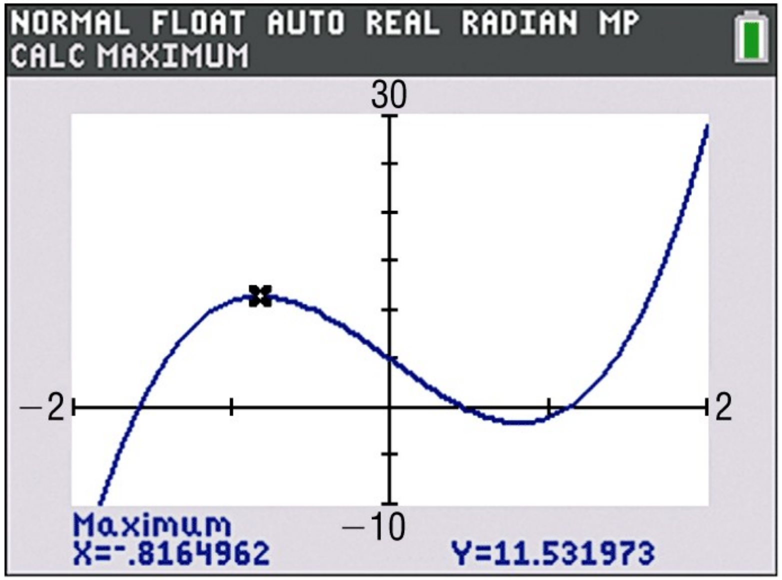
(b) Determine where f is increasing and where it is decreasing.

Solution

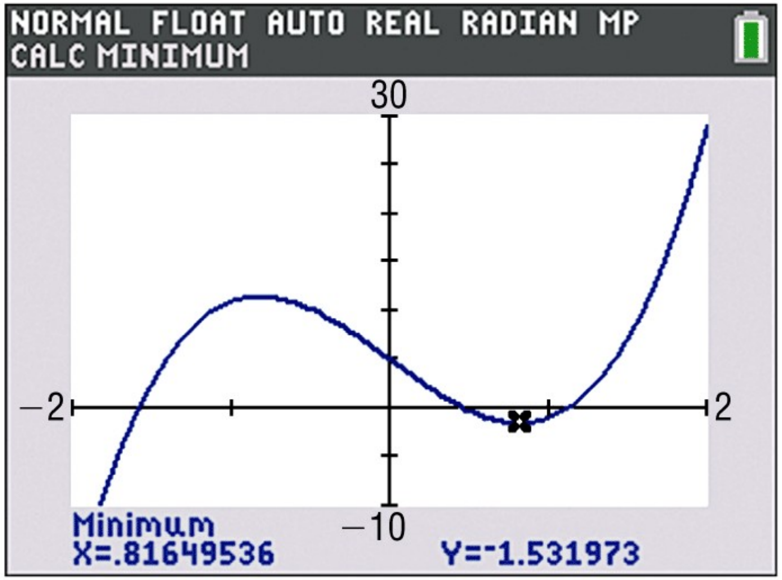
(a) Graphing utilities have a feature that finds the maximum or minimum point of a graph within a given interval. Graph the function f for $-2 < x < 2$. The MAXIMUM and MINIMUM commands require us to first determine the open interval I. The graphing utility will then approximate the maximum or minimum value in the interval. Using MAXIMUM, we find that the local maximum value is 11.53 and that it occurs at $x = -0.82$, rounded to two decimal places. See Figure 24(a). Using MINIMUM, we find that the local minimum value is -1.53 and that it occurs at $x = 0.82$, rounded to two decimal places. See Figure 24(b).

(b) Looking at Figures 24(a) and (b), we see that the graph of f is increasing from $x = -2$ to $x = -0.82$ and from $x = 0.82$ to $x = 2$, so f is increasing on the intervals $(-2, -0.82)$ and $(0.82, 2)$, or for $-2 < x < -0.82$ and $0.82 < x < 2$. The graph is decreasing from $x = -0.82$ to $x = 0.82$, so f is decreasing on the interval $(-0.82, 0.82)$, or for $-0.82 < x < 0.82$.

Solution continued



(a) Local maximum



(b) Local minimum

Figure 24

Find the Average Rate of Change of a Function

Definition

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f** from a to b is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad \textbf{(1)}$$

Example

Finding the Average Rate of Change

Find the average rate of change of $f(x) = 3x^2$:

(a) From 1 to 3 (b) From 1 to 5 (c) From 1 to 7

Solution

(a) The average rate of change of $f(x) = 3x^2$ from 1 to 3 is

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 3}{3 - 1} = \frac{24}{2} = 12$$

(b) The average rate of change of $f(x) = 3x^2$ from 1 to 5 is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(1)}{5 - 1} = \frac{75 - 3}{5 - 1} = \frac{72}{4} = 18$$

(c) The average rate of change of $f(x) = 3x^2$ from 1 to 7 is

$$\frac{\Delta y}{\Delta x} = \frac{f(7) - f(1)}{7 - 1} = \frac{147 - 3}{7 - 1} = \frac{144}{6} = 24$$

Figure: $f(x) = 3x^2$

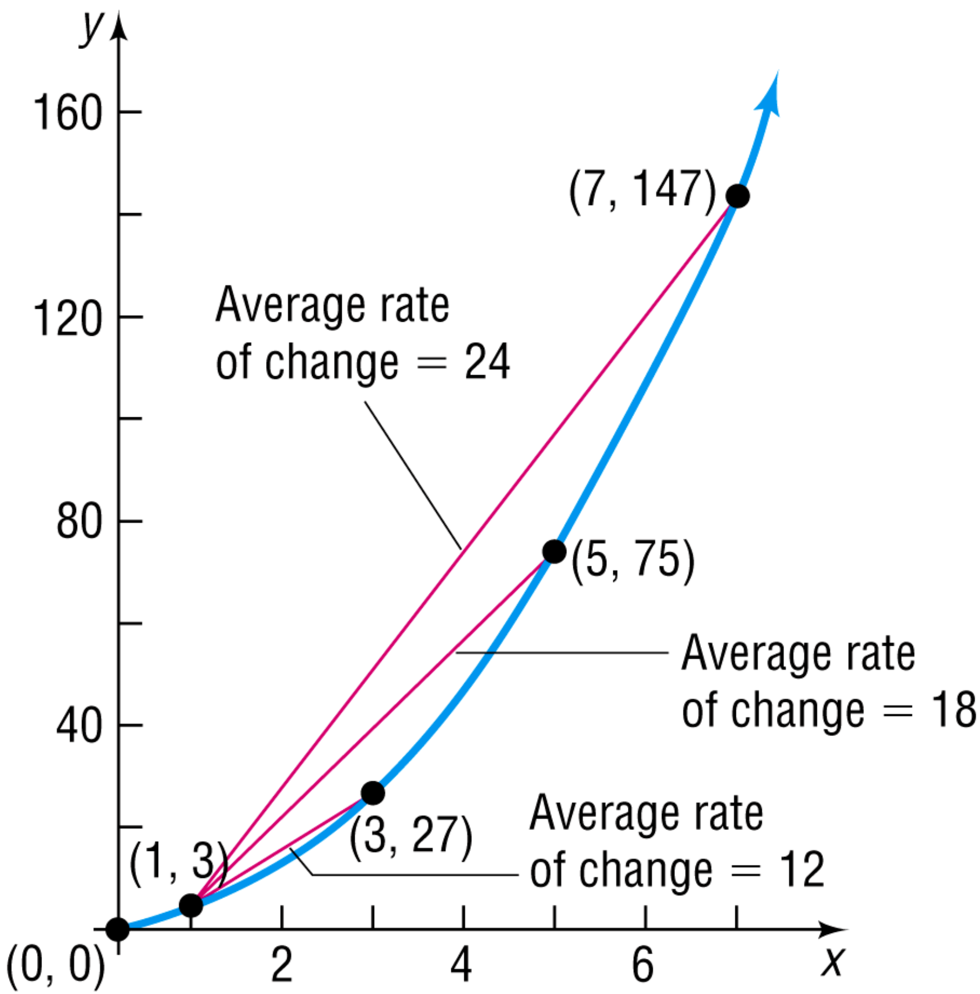
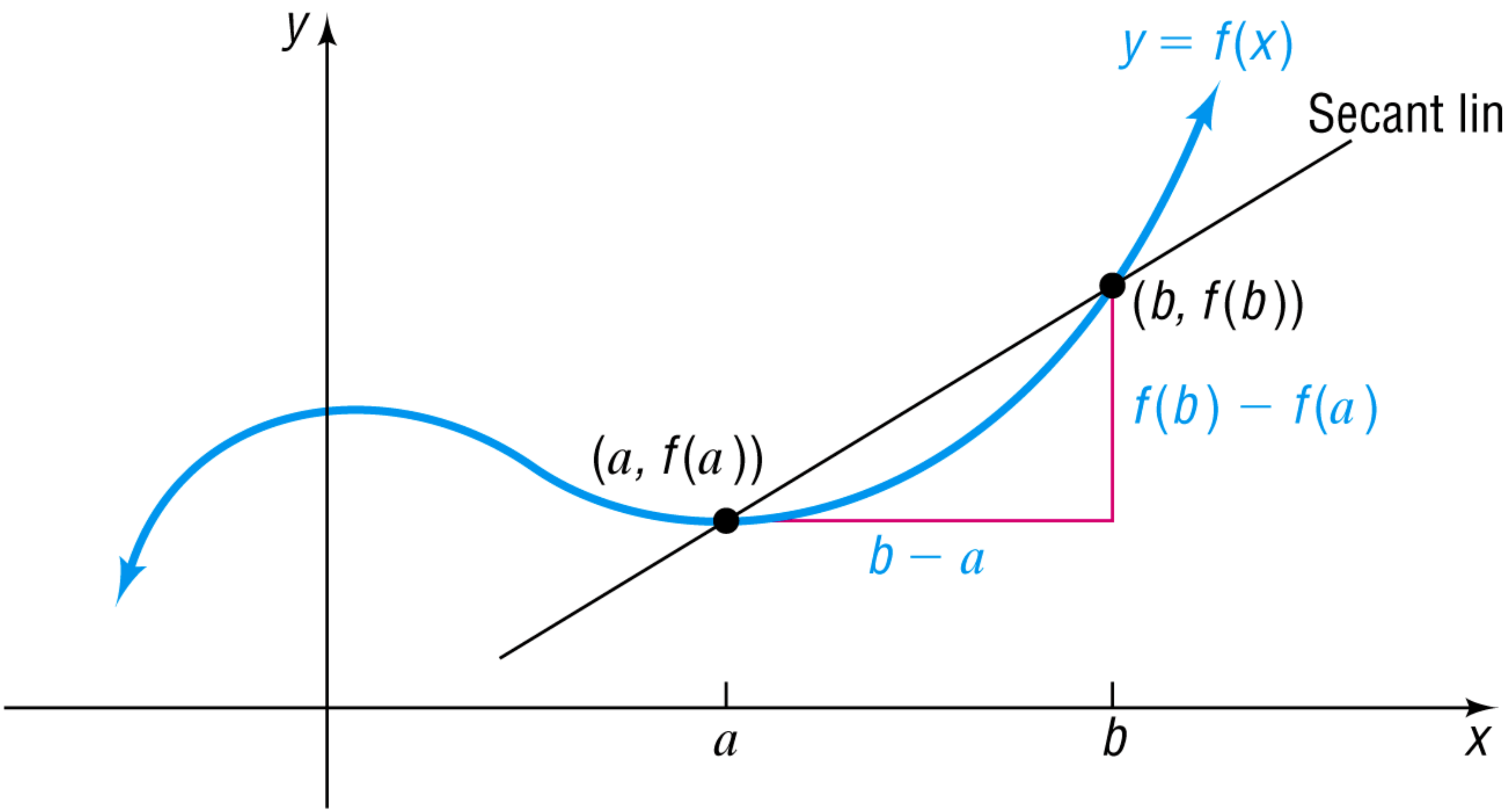


Figure: Secant Line



Theorem

Slope of the Secant Line

The average rate of change of a function from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

Example

Finding the Equation of a Secant Line

Suppose that $g(x) = 3x^2 + 2x - 1.$

(a) Find the average rate of change of g from –2 to 1.

(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

(c) Draw the graph of g and the secant line obtained in part (b) on the same axes.

Solution

(a) The average rate of change of $g(x) = 3x^2 + 2x - 1$ from –2 to 1 is

$$\text{Average rate of change} = \frac{g(1) - g(-2)}{1 - (-2)}$$

$$= \frac{4 - 7}{3} \qquad g(1) = 3(1)^2 + 2(1) - 1 = 4$$

$$g(-2) = 3(-2)^2 + 2(-2) - 1 = 7$$

$$= -\frac{3}{3} = -1$$

Solution continued

(b) The slope of the secant line containing $\left(-2, g(-2)\right) = \left(-2, 7\right)$ and $\left(1, g(1)\right) = \left(1, 4\right)$ is $m_{\text{sec}} = -1.$ Use the point-slope form to find an equation of the secant line.

$y - y_1 = m_{\text{sec}}\left(x - x_1\right)$	Point-slope form of the secant line
$y - 7 = -1\left(x - (-2)\right)$	$x_1 = -2, y_1 = g(-2) = 7, m_{\text{sec}} = -1$
$y - 7 = -x - 2$	Distribute.
$y = -x + 5$	Slope-intercept form of the secant line

Solution continued

(c) The figure shows the graph of g along with the secant line $y = -x + 5$.