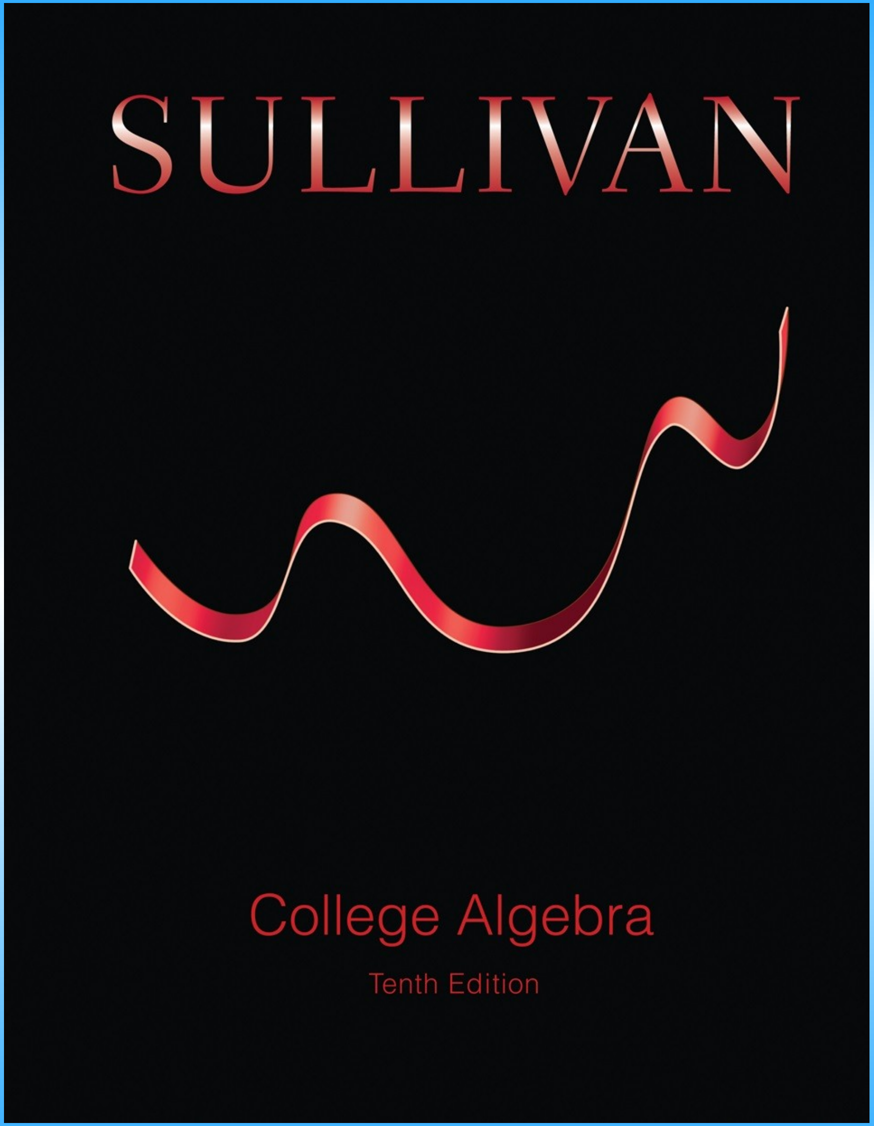
Chapter R

Section 7



R.7 Rational Expressions

OBJECTIVES

Reduce a Rational Expression to Lowest Terms

Example

Reducing a Rational Expression to Lowest Terms

Reduce to lowest terms: $\dfrac{x^2 + 4x + 4}{x^2 + 3x + 2}$

Solution

Begin by factoring the numerator and the denominator.

$$x^2 + 4x + 4 = (x + 2)(x + 2)$$

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

Since a common factor, $x + 2$, appears, the original expression is not in lowest terms. To reduce it to lowest terms, use the Cancellation Property:

$$\frac{x^2 + 4x + 4}{x^2 + 3x + 2} = \frac{\cancel{(x + 2)}(x + 2)}{\cancel{(x + 2)}(x + 1)} = \frac{x + 2}{x + 1} \qquad x \neq -2, -1$$

Multiply and Divide Rational Expressions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(2)}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{if } b \neq 0, c \neq 0, d \neq 0 \qquad \textbf{(3)}$$

Example

Multiplying and Dividing Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x^2 - 2x + 1}{x^3 + x} \cdot \dfrac{4x^2 + 4}{x^2 + x - 2}$

(b) $\dfrac{\dfrac{x + 3}{x^2 - 4}}{\dfrac{x^2 - x - 12}{x^3 - 8}}$

Solution

(a) $$\frac{x^2 - 2x + 1}{x^3 + x} \cdot \frac{4x^2 + 4}{x^2 + x - 2} = \frac{(x-1)^2}{x(x^2+1)} \cdot \frac{4(x^2+1)}{(x+2)(x-1)}$$

$$= \frac{(x-1)^{2}(4)(x^2+1)}{x(x^2+1)(x+2)(x-1)}$$

$$= \frac{4(x-1)}{x(x+2)} \qquad x \neq -2, 0, 1$$

(b) $$\frac{\dfrac{x+3}{x^2-4}}{\dfrac{x^2-x-12}{x^3-8}} = \frac{x+3}{x^2-4} \cdot \frac{x^3-8}{x^2-x-12}$$

$$= \frac{x+3}{(x-2)(x+2)} \cdot \frac{(x-2)(x^2+2x+4)}{(x-4)(x+3)}$$

$$= \frac{(x+3)(x-2)(x^2+2x+4)}{(x-2)(x+2)(x-4)(x+3)}$$

$$= \frac{x^2+2x+4}{(x+2)(x-4)} \qquad x \neq -3, -2, 2, 4$$

Add and Subtract Rational Expressions

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b} \qquad \text{if } b \neq 0 \qquad \textbf{(4)}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{ad + bc}{bd} \quad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(5a)}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d} = \frac{ad - bc}{bd} \quad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(5b)}$$

Example

Adding and Subtracting Rational Expressions with Unequal Denominators

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x-3}{x+4} + \dfrac{x}{x-2} \qquad x \neq -4, 2$

(b) $\dfrac{x^2}{x^2-4} - \dfrac{1}{x} \qquad x \neq -2, 0, 2$

Solution

$$\text{(a)}\quad \frac{x-3}{x+4}+\frac{x}{x-2}=\frac{x-3}{x+4}\cdot\frac{x-2}{x-2}+\frac{x+4}{x+4}\cdot\frac{x}{x-2}$$

(5a)

$$=\frac{(x-3)(x-2)+(x+4)(x)}{(x+4)(x-2)}$$

$$=\frac{x^2-5x+6+x^2+4x}{(x+4)(x-2)}=\frac{2x^2-x+6}{(x+4)(x-2)}$$

Solution continued

$$\text{(b)}\ \frac{x^2}{x^2-4}-\frac{1}{x}=\frac{x^2}{x^2-4}\cdot\frac{x}{x}-\frac{x^2-4}{x^2-4}\cdot\frac{1}{x}=\frac{x^2(x)-(x^2-4)(1)}{(x^2-4)(x)}$$

(5b)

$$=\frac{x^3-x^2+4}{(x-2)(x+2)(x)}$$

Use the Least Common Multiple Method

The LCM Method for Adding or Subtracting Rational Expressions

The Least Common Multiple (LCM) Method requires four steps:

STEP 1: Factor completely the polynomial in the denominator of each rational expression.

STEP 2: The LCM of the denominators is the product of each of these factors raised to a power equal to the greatest number of times that the factor occurs in the polynomials.

STEP 3: Write each rational expression using the LCM as the common denominator.

STEP 4: Add or subtract the rational expressions using equation (4).

Example

Using the Least Common Multiple to Add Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{x}{x^2 + 3x + 2} + \frac{2x - 3}{x^2 - 1} \qquad x \neq -2, -1, 1$$

Solution

STEP 1: Factor completely the polynomials in the denominators.

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$
$$x^2 - 1 = (x - 1)(x + 1)$$

STEP 2: The LCM is $(x + 2)(x + 1)(x - 1)$. Do you see why?

STEP 3: Write each rational expression using the LCM as the denominator.

$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x + 2)(x + 1)} = \frac{x}{(x + 2)(x + 1)} \cdot \frac{x - 1}{x - 1} = \frac{x(x - 1)}{(x + 2)(x + 1)(x - 1)}$$

↑ Multiply numerator and denominator by $x - 1$ to get the LCM in the denominator.

Solution continued

$$\frac{2x-3}{x^2-1} = \frac{2x-3}{(x-1)(x+1)} = \frac{2x-3}{(x-1)(x+1)} \cdot \frac{x+2}{x+2} = \frac{(2x-3)(x+2)}{(x-1)(x+1)(x+2)}$$

Multiply numerator and denominator by $x + 2$ to get the LCM in the denominator.

Step 4: Now add by using equation (4).

$$\frac{x}{x^2+3x+2} + \frac{2x-3}{x^2-1} = \frac{x(x-1)}{(x+2)(x+1)(x-1)} + \frac{(2x-3)(x+2)}{(x+2)(x+1)(x-1)}$$

$$= \frac{(x^2-x) + (2x^2+x-6)}{(x+2)(x+1)(x-1)}$$

$$= \frac{3x^2-6}{(x+2)(x+1)(x-1)} = \frac{3(x^2-2)}{(x+2)(x+1)(x-1)}$$

Example

Using the Least Common Multiple to Subtract Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{3}{x^2 + x} - \frac{x + 4}{x^2 + 2x + 1} \qquad x \neq -1, 0$$

Solution

Step 1: Factor completely the polynomials in the denominators.

$$x^2 + x = x(x + 1)$$

$$x^2 + 2x + 1 = (x + 1)^2$$

Step 2: The LCM is $x(x + 1)^2$.

Solution continued

Step 3: Write each rational expression using the LCM as the denominator.

$$\frac{3}{x^2+x} = \frac{3}{x(x+1)} = \frac{3}{x(x+1)} \cdot \frac{x+1}{x+1} = \frac{3(x+1)}{x(x+1)^2}$$

$$\frac{x+4}{x^2+2x+1} = \frac{x+4}{(x+1)^2} = \frac{x+4}{(x+1)^2} \cdot \frac{x}{x} = \frac{x(x+4)}{x(x+1)^2}$$

Step 4: Subtract, using equation (4).

$$\begin{aligned} \frac{3}{x^2+x} - \frac{x+4}{x^2+2x+1} &= \frac{3(x+1)}{x(x+1)^2} - \frac{x(x+4)}{x(x+1)^2} \\ &= \frac{3(x+1) - x(x+4)}{x(x+1)^2} \\ &= \frac{3x+3-x^2-4x}{x(x+1)^2} \\ &= \frac{-x^2-x+3}{x(x+1)^2} \end{aligned}$$

Simplify Complex Rational Expressions

Simplifying a Complex Rational Expression

Option 1: Treat the numerator and denominator of the complex rational expression separately, performing whatever operations are indicated and simplifying the results. Follow this by simplifying the resulting rational expression.

Option 2: Find the LCM of the denominators of all rational expressions that appear in the complex rational expression. Multiply the numerator and denominator of the complex rational expression by the LCM and simplify the result.

Example

Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{x^2}{x-4}+2}{\dfrac{2x-2}{x}-1} \qquad x \neq 0, 2, 4$

Solution

We will use Option 1.

$$\frac{\frac{x^2}{x-4}+2}{\frac{2x-2}{x}-1} = \frac{\frac{x^2}{x-4}+\frac{2(x-4)}{x-4}}{\frac{2x-2}{x}-\frac{x}{x}} = \frac{\frac{x^2+2x-8}{x-4}}{\frac{2x-2-x}{x}}$$

$$= \frac{\frac{(x+4)(x-2)}{x-4}}{\frac{x-2}{x}} = \frac{(x+4)\cancel{(x-2)}}{x-4}\cdot\frac{x}{\cancel{x-2}}$$

$$= \frac{(x+4)\cdot x}{x-4}$$

Example

Solving an Application in Electricity

An electrical circuit contains two resistors connected in parallel, as shown in Figure 28. If these two resistors provide resistance of R_1 and R_2 ohms, respectively, their combined resistance R is given by the formula

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Express R as a rational expression; that is, simplify the right-hand side of this formula. Evaluate the rational expression if $R_1 = 6$ ohms and $R_2 = 10$ ohms.

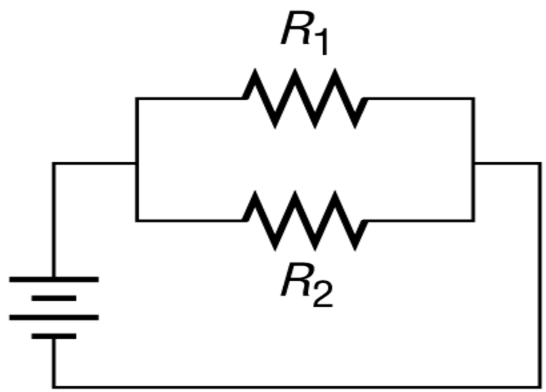


Figure 28

Solution

We will use Option 2. If we consider 1 as the fraction $\frac{1}{1}$, the rational expressions in the complex rational expression are

$$\frac{1}{1}, \quad \frac{1}{R_1}, \quad \frac{1}{R_2}$$

The LCM of the denominators is $R_1 R_2$. We multiply the numerator and denominator of the complex rational expression by $R_1 R_2$ and simplify.

$$\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1 \cdot R_1 R_2}{\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \cdot R_1 R_2} = \frac{R_1 R_2}{\frac{1}{R_1} \cdot R_1 R_2 + \frac{1}{R_2} \cdot R_1 R_2} = \frac{R_1 R_2}{R_2 + R_1}$$

So,

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

If $R_1 = 6$ and $R_2 = 10$, then

$$R = \frac{6 \cdot 10}{10 + 6} = \frac{60}{16} = \frac{15}{4} \quad \text{ohms}$$