

The Real Numbers and Their Representation



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Chapter 6: The Real Numbers and Their Representation

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Section 6-3

Rational Numbers and Decimal Representation

Rational Numbers and Decimal Representation

- Define and identify rational numbers.
- Write a rational number in lowest terms.
- Add, subtract, multiply, and divide rational numbers in fraction form.
- Solve carpentry problem using operations with fractions.
- Apply the density property and find arithmetic mean.

Rational Numbers and Decimal Representation

- Convert a rational number in fraction form to a decimal number.
- Convert a terminating or repeating decimal to a rational number in fraction form.

Definition: Rational Numbers

Rational Numbers =

$\{x \mid x \text{ is a quotient of two integers, with denominator not equal to } 0\}$

Lowest Terms

A rational number is said to be in **lowest terms** if the greatest common factor of the numerator (top number) and the denominator (bottom number) is 1. Rational numbers are written in lowest terms by using the *fundamental property of rational numbers* (see next slide).

Fundamental Property of Rational Numbers

If a , b , and k are integers with $b \neq 0$ and $k \neq 0$, then

$$\frac{a \cdot k}{b \cdot k} = \frac{a}{b}.$$

Example: Writing a Fraction in Lowest Terms

Write $\frac{24}{27}$ in lowest terms.

Solution

$$\frac{24}{27} = \frac{8 \cdot 3}{9 \cdot 3} = \frac{8}{9}.$$

Cross-Product Test for Equality of Rational Numbers

For rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } a \cdot d = b \cdot c.$$

a and d are called the “**extremes.**”

b and c are called the “**means.**”

Adding and Subtracting Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Adding and Subtracting Rational Numbers

In practical problems involving addition and subtraction of rational numbers, fractions are usually rewritten with the least common multiple of their denominators, called the **least common denominator**.

Example: Adding and Subtracting Rational Numbers

Perform each operation.

$$\text{a) } \frac{4}{9} + \frac{2}{15}$$

$$\text{b) } \frac{4}{9} - \frac{2}{15}$$

Solution

$$\text{a) } \frac{4}{9} + \frac{2}{15} = \frac{20}{45} + \frac{6}{45} = \frac{26}{45}$$

$$\text{b) } \frac{4}{9} - \frac{2}{15} = \frac{20}{45} - \frac{6}{45} = \frac{14}{45}$$

Multiplying Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Example: Multiplying Rational Numbers

Find the product $\frac{5}{9} \cdot \frac{3}{10}$.

Solution

$$\frac{5}{9} \cdot \frac{3}{10} = \frac{15}{90} = \frac{1 \cdot 15}{6 \cdot 15} = \frac{1}{6}$$

Definition of Division

If a and b are real numbers with $b \neq 0$, then

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

Dividing Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers $\left(\frac{c}{d} \neq 0\right)$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example: Dividing Rational Numbers

Find the quotient $\frac{2}{9} \div \frac{5}{6}$.

Solution

$$\frac{2}{9} \div \frac{5}{6} = \frac{2}{9} \cdot \frac{6}{5} = \frac{12}{45} = \frac{4 \cdot 3}{15 \cdot 3} = \frac{4}{15}$$

Example: Applying Operations with Fractions to Carpentry

A carpenter has a board 72 inches long that he must cut into 10 pieces of equal length. This will require 9 cuts.

a) If each cut causes a waste of $\frac{3}{16}$ inch, how many inches of actual board will remain after the cuts?

b) What will the length of each of the resulting pieces be?

Continued

Example: Applying Operations with Fractions to Carpentry

Solution

a) We start with 72 inches and must subtract $\frac{3}{16}$ nine times. We represent this by $9\left(\frac{3}{16}\right)$.

$$\begin{aligned} 72 - 9\left(\frac{3}{16}\right) &= 72 - \left(\frac{9}{1} \square \frac{3}{16}\right) \\ &= 72 - \frac{27}{16} = \frac{1125}{16} = 70\frac{5}{16} \end{aligned}$$

Thus, $70\frac{5}{16}$ inches of actual board will remain.

Continued

Example: Applying Operations with Fractions to Carpentry

Solution

b) The $70\frac{5}{16}$ or $\frac{1125}{16}$ inches of board from part (a) will be divided into 10 pieces of equal length.

$$\frac{1125}{16} \div 10 = \frac{1125}{16} \cdot \frac{1}{10}$$

$$= \frac{225}{32} = 7\frac{1}{32}$$

Each piece will measure $7\frac{1}{32}$ inches.

Density Property of Rational Numbers

If r and t are distinct rational numbers, with $r < t$, then there exists a rational number s such that

$$r < s < t.$$

This leads to the conclusion that there are *infinitely many* rational numbers between two distinct rational numbers.

Arithmetic Mean



To find the **arithmetic mean**, or **average**, of n numbers, we add the numbers and then divide the sum by n . For two numbers, the number that lies halfway between them is their average.

Example: Finding an Arithmetic Mean (Average)

Find the average of $\frac{3}{5}$ and $\frac{2}{3}$.

Solution

$$\frac{3}{5} + \frac{2}{3} = \frac{9}{15} + \frac{10}{15} = \frac{19}{15}$$

Add the fractions.

$$\frac{19}{15} \div 2 = \frac{19}{15} \cdot \frac{1}{2} = \frac{19}{30}$$

Divide the sum by 2 to get the answer.

Converting a Rational Number in Fraction Form to a Decimal Number

A rational number in the form a/b can be expressed as a decimal most easily by entering it into a calculator.

For example, to write $\frac{7}{8}$ as a decimal, enter 7, enter the operation of division, enter 8, and then press the equals key to find the following equivalence:

$$\frac{7}{8} = 0.875$$

Decimal Form of Rational Numbers

Any rational number can be expressed as either a *terminating* (stops) decimal or a *repeating* (reoccurring block of numbers) decimal.

Criteria for Terminating and Repeating Decimals

A rational number, in lowest terms, results in a **terminating decimal** if the only prime factor of the denominator is 2 or 5 (or both).

A rational number, in lowest terms, results in a **repeating decimal** if a prime other than 2 or 5 appears in the prime factorization of the denominator.

Example: Determining Whether a Decimal Terminates or Repeats

Without dividing, determine whether the decimal form of the given rational number terminates or repeats.

a) $\frac{7}{15}$

b) $\frac{15}{16}$

Solution

- a) Repeats; there is a factor of 3 in the denominator.
- b) Terminates; the denominator is 2^4 .

Example: Writing Decimals as Quotients of Integers

Write each decimal as a quotient of integers.

a) 0.759

b) 7.8

Solution

$$\text{a) } 0.759 = \frac{759}{1000}$$

$$\text{b) } 7.8 = 7 + \frac{8}{10} = \frac{78}{10} = \frac{39}{5}$$