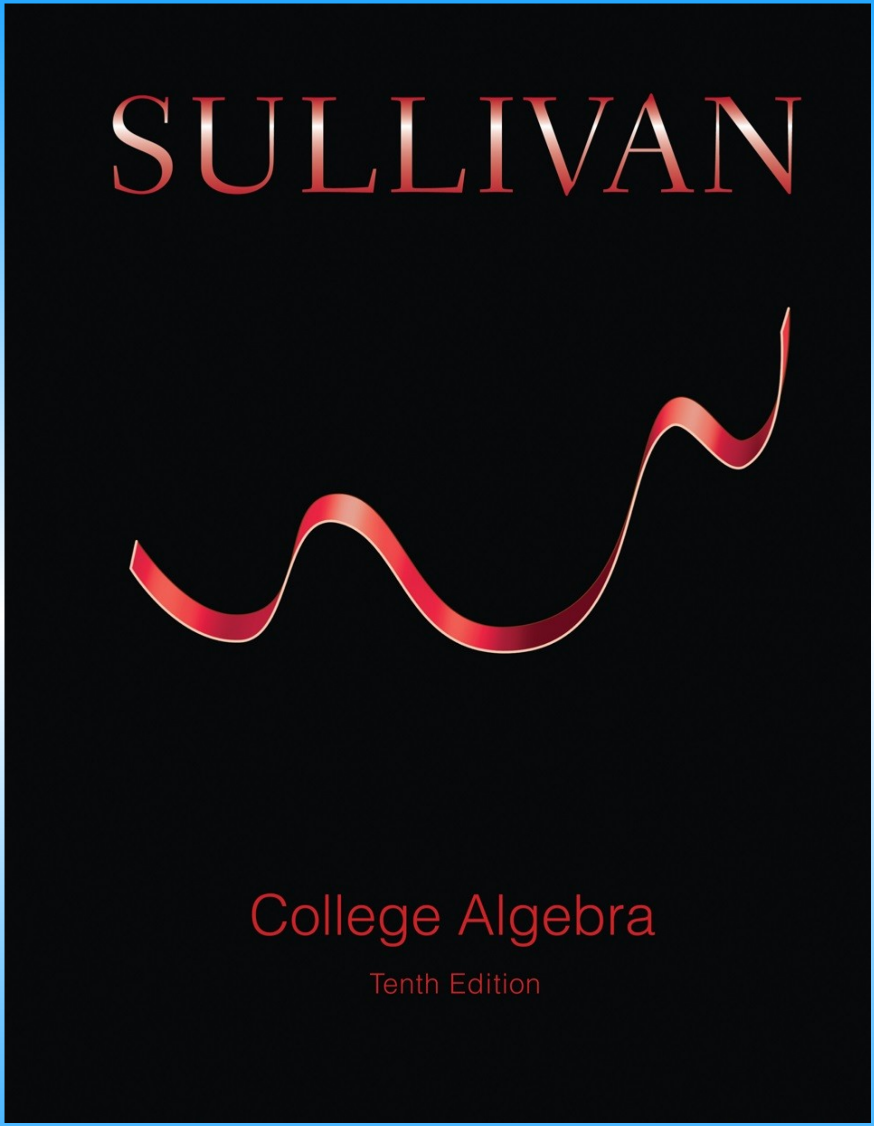
Chapter 1

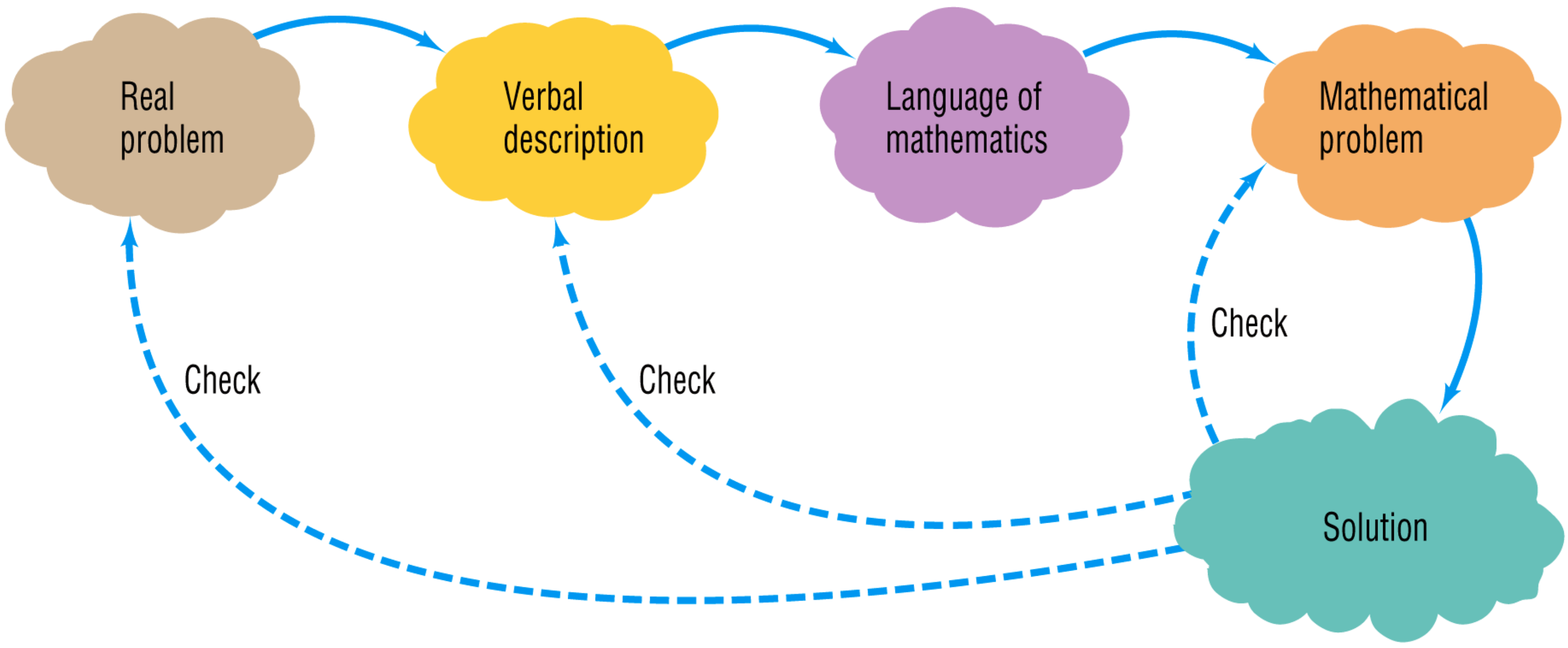
Section 7



1.7 Problem Solving: Interest, Mixture, Uniform Motion, Constant Rate Job Applications

OBJECTIVES

The Modeling Process



Translate Verbal Descriptions into Mathematical Expressions

Example

Translating Verbal Descriptions into Mathematical Expressions

(a) For uniform motion, the average speed of an object equals the distance traveled divided by the time required.

Translation: If r is the speed, d the distance, and t the time, then $r = \frac{d}{t}$.

(b) Let x denote a number.

The number 5 times as large as x is $5x$.

The number 3 less than x is $x - 3$.

The number that exceeds x by 4 is $x + 4$.

The number that, when added to x, gives 5 is $5 - x$.

Steps for Solving Applied Problems

STEP 1: Read the problem carefully, perhaps two or three times. Pay particular attention to the question being asked in order to identify what you are looking for. Identify any relevant formulas you may need ($d = rt$, $A = \pi r^2$, etc.). If you can, determine realistic possibilities for the answer.

STEP 2: Assign a letter (variable) to represent what you are looking for, and if necessary, express any remaining unknown quantities in terms of this variable.

STEP 3: Make a list of all the known facts, and translate them into mathematical expressions. These may take the form of an equation or an inequality involving the variable. If possible, draw an appropriately labeled diagram to assist you. Sometimes, creating a table or chart helps.

STEP 4: Solve for the variable, and then answer the question.

STEP 5: Check the answer with the facts in the problem. If it agrees, congratulations! If it does not agree, try again.

Solve Interest Problems

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt \qquad \textbf{(1)}$$

Example

Financial Planning

Candy has $70,000 to invest and wants an annual return of $2800, which requires an overall rate of return of 4%. She can invest in a safe, government-insured certificate of deposit, but it pays only 2%. To obtain 4%, she agrees to invest some of her money in noninsured corporate bonds paying 7%. How much should be placed in each investment to achieve her goal?

Solution

Step 1: The question is asking for two dollar amounts: the principal to invest in the corporate bonds and the principal to invest in the certificate of deposit.

Step 2: Let b represent the amount (in dollars) to be invested in the bonds. Then $70{,}000 - b$ is the amount that will be invested in the certificate. (Do you see why?)

Step 3: We set up a table:

	Principal ($)	Rate	Time (yr)	Interest ($)
Bonds	b	$7\% = 0.07$	1	$0.07b$
Certificate	$70{,}000 - b$	$2\% = 0.02$	1	$0.02(70{,}000 - b)$
Total	70,000	$4\% = 0.04$	1	$0.04(70{,}000) = 2800$

Solution continued

Since the combined interest from the investments is equal to the total interest, we have

$$\text{Bond interest} + \text{Certificate interest} = \text{Total interest}$$
$$0.07b + 0.02(70{,}000 - b) = 2800$$

(Note that the units are consistent: the unit is dollars on each side.)

STEP 4: $0.07b + 1400 - 0.02b = 2800$

$$0.05b = 1400$$
$$b = 28{,}000$$

Candy should place \$28,000 in the bonds and \$70,000 − \$28,000 = \$42,000 in the certificate.

✓ **STEP 5:** The interest on the bonds after 1 year is 0.07(\$28,000) = \$1960; the interest on the certificate after 1 year is 0.02(\$42,000) = \$840. The total annual interest is \$2800, the required amount.

Solve Mixture Problems

Example

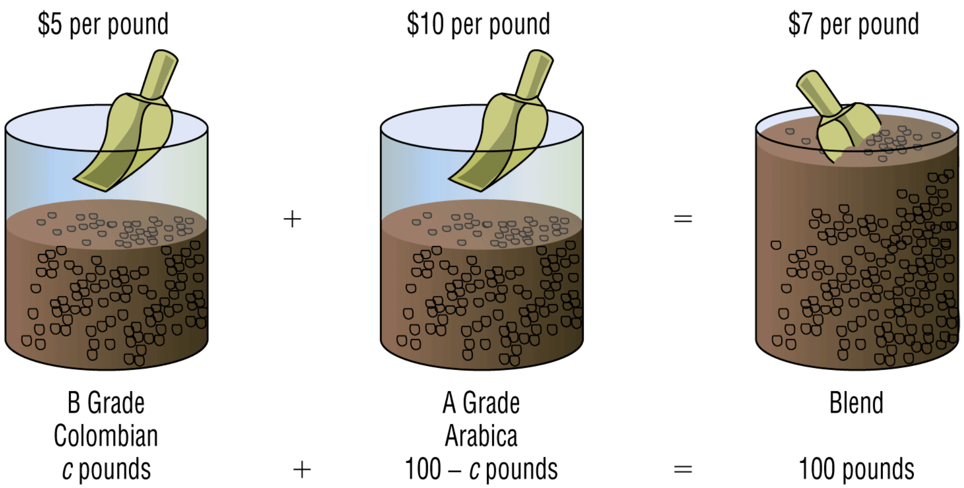
Blending Coffees

The manager of a Starbucks store decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for $5 per pound with some A grade Arabica coffee that sells for $10 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be $7 per pound, and there is to be no difference in revenue between selling the new blend and selling the other types. How many pounds of the B grade Colombian coffee and how many pounds of the A grade Arabica coffees are required?

Solution

Let c represent the number of pounds of the B grade Colombian coffee. Then $100 - c$ equals the number of pounds of the A grade Arabica coffee. See Figure 16.

Figure 16



Since there is to be no difference in revenue between selling the A and B grades separately and selling the blend, we have

$$\text{Revenue from B grade} \quad + \quad \text{Revenue from A grade} \quad = \quad \text{Revenue from blend}$$

$$\left\{\begin{matrix}\text{Price per pound}\\ \text{of B grade}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{B grade}\end{matrix}\right\} + \left\{\begin{matrix}\text{Price per pound}\\ \text{of A grade}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{A grade}\end{matrix}\right\} = \left\{\begin{matrix}\text{Price per pound}\\ \text{of blend}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{blend}\end{matrix}\right\}$$

$$\$5 \cdot c + \$10 \cdot (100 - c) = \$7 \cdot 100$$

Solution continued

Now solve the equation:

$$
\begin{aligned}
5c + 10(100 - c) &= 700 \\
5c + 1000 - 10c &= 700 \\
-5c &= -300 \\
c &= 60
\end{aligned}
$$

The manager should blend 60 pounds of B grade Colombian coffee with $100 - 60 = 40$ pounds of A grade Arabica coffee to get the desired blend.

✓**Check:** The 60 pounds of B grade coffee would sell for $(\$5)(60) = \$300$, and the 40 pounds of A grade coffee would sell for $(\$10)(40) = \$400$; the total revenue, \$700, equals the revenue obtained from selling the blend, as desired.

Solve Uniform Motion Problems

Uniform Motion Formula

If an object moves at an average speed (rate) r, the distance d covered in time t is given by the formula

$$d = rt \qquad \textbf{(2)}$$

That is, Distance = Rate · Time.

Example

Physics: Uniform Motion

Tanya, who is a long-distance runner, runs at an average speed of 8 miles per hour (mi/h). Two hours after Tanya leaves your house, you leave in your Honda and follow the same route. If your average speed is 40 mi/h, how long will it be before you catch up to Tanya? How far will each of you be from your home?

Solution

Refer to Figure 17. We use t to represent the time (in hours) that it takes the Honda to catch up to Tanya. When this occurs, the total time elapsed for Tanya is $t + 2$ hours because she left 2 hours earlier.

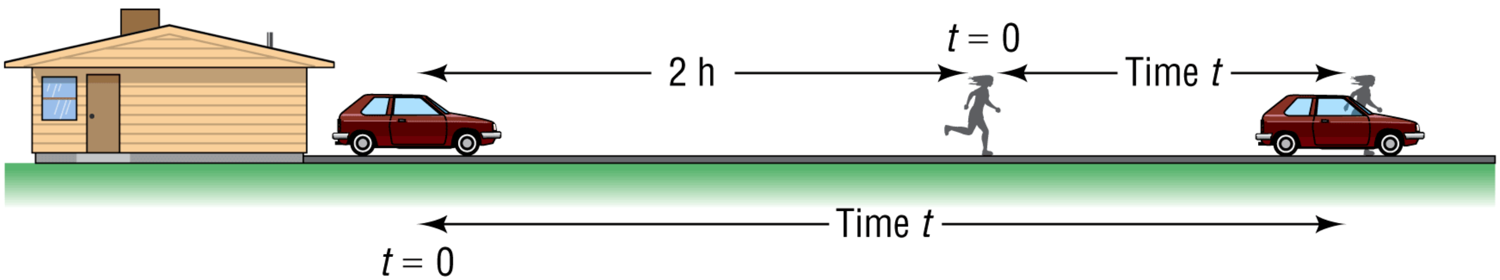


Figure 17

Set up the following table:

	Rate mi/h	Time h	Distance mi
Tanya	8	$t + 2$	$8(t + 2)$
Honda	40	t	$40t$

Solution continued

The distance traveled is the same for both, which leads to the equation

$$8(t + 2) = 40t$$
$$8t + 16 = 40t$$
$$32t = 16$$
$$t = \frac{1}{2} \text{ hour}$$

It will take the Honda $\frac{1}{2}$ hour to catch up to Tanya. Each will have gone 20 miles.

✓**Check:** In 2.5 hours, Tanya travels a distance of $(2.5)(8) = 20$ miles. In $\frac{1}{2}$ hour, the Honda travels a distance of $\left(\frac{1}{2}\right)(40) = 20$ miles.

Example

Physics: Uniform Motion

A motorboat heads upstream a distance of 24 miles on a river whose current is running at 3 miles per hour (mi/h). The trip up and back takes 6 hours. Assuming that the motorboat maintained a constant speed relative to the water, what was its speed?

Solution

See Figure 18. Use r to represent the constant speed of the motorboat relative to the water. Then the true speed going upstream is $r - 3$ mi/h, and the true speed going downstream is $r + 3$ mi/h. Since Distance = Rate · Time, then Time $= \frac{\text{Distance}}{\text{Rate}}$. Set up a table.

	Rate mi/h	Distance mi	Time h $= \frac{\text{Distance}}{\text{Rate}}$
Upstream	$r - 3$	24	$\frac{24}{r - 3}$
Downstream	$r + 3$	24	$\frac{24}{r + 3}$

Solution continued

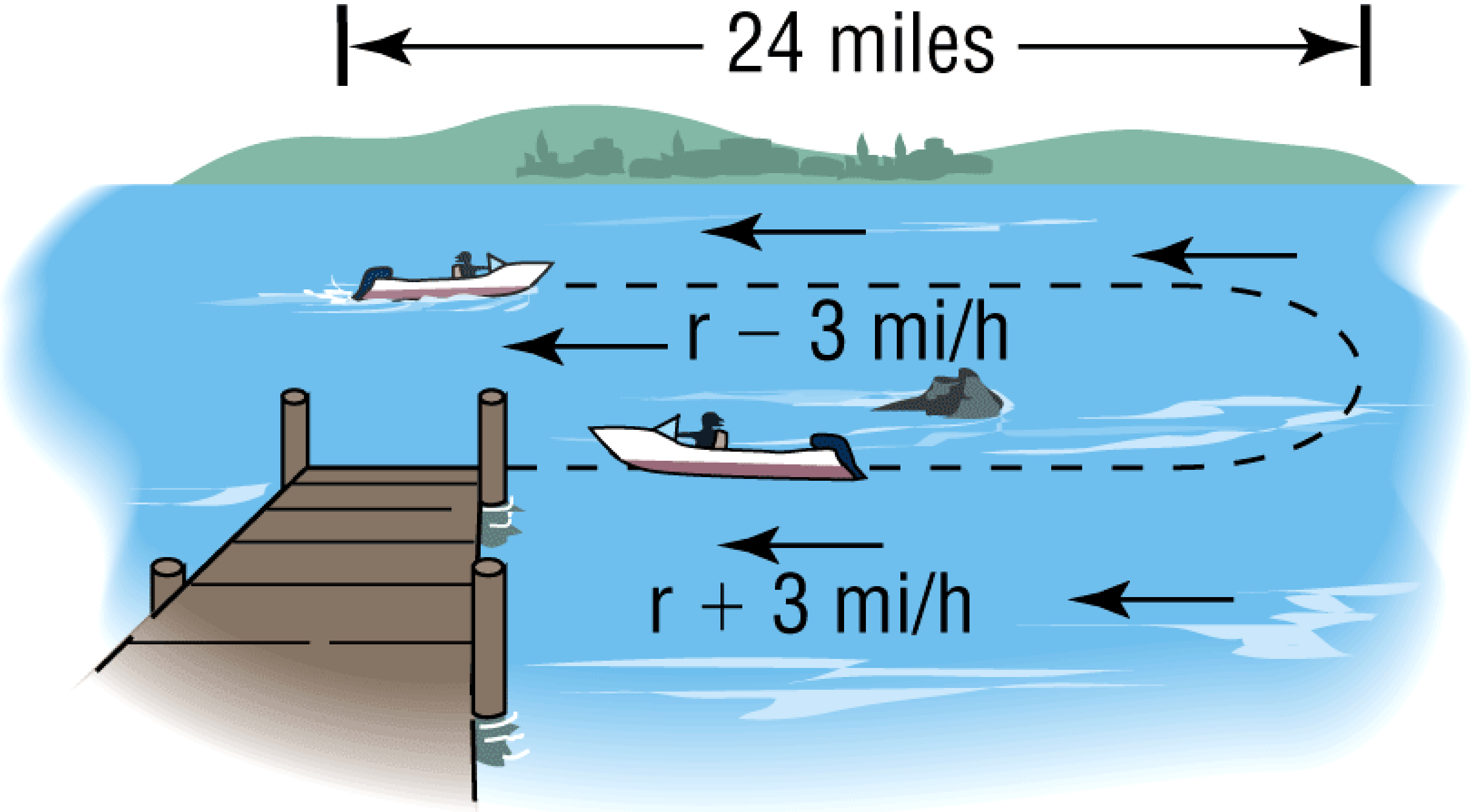


Figure 18

Solution continued

The total time up and back is 6 hours, which gives the equation

$$\frac{24}{r-3} + \frac{24}{r+3} = 6$$

$$\frac{24(r+3) + 24(r-3)}{(r-3)(r+3)} = 6 \qquad \text{Add the quotients on the left.}$$

$$\frac{48r}{r^2-9} = 6 \qquad \text{Simplify.}$$

$$48r = 6(r^2-9) \qquad \text{Multiply both sides by } r^2 - 9.$$

$$6r^2 - 48r - 54 = 0 \qquad \text{Place in standard form.}$$

$$r^2 - 8r - 9 = 0 \qquad \text{Divide by 6.}$$

$$(r-9)(r+1) = 0 \qquad \text{Factor.}$$

$$r = 9 \quad \text{or} \quad r = -1 \qquad \text{Apply the Zero-Product Property and solve.}$$

Discard the solution $r = -1$ mi/h and conclude that the speed of the motorboat relative to the water is 9 mi/h.

Solve Constant Rate Job Problems

Example

Working Together to Do a Job

At 10 am Danny is asked by his father to weed the garden. From past experience, Danny knows that this will take him 5 hours, working alone. His older brother Mike, when it is his turn to do this job, requires 6 hours. Since Mike wants to go golfing with Danny and has a reservation for 1 pm, he agrees to help Danny. Assuming no gain or loss of efficiency, when will they finish if they work together? Can they make the golf date?

Solution

Set up a table. In 1 hour, Danny does $1/5$ of the job, and in 1 hour, Mike does $1/6$ of the job.

Let t be the time (in hours) that it takes them to do the job together. In 1 hour then, $1/t$ of the job is completed. Reason as follows:

$$\begin{pmatrix}\text{Part done}\\ \text{by Danny}\\ \text{in 1 hour}\end{pmatrix} + \begin{pmatrix}\text{Part done}\\ \text{by Mike}\\ \text{in 1 hour}\end{pmatrix} = \begin{pmatrix}\text{Part done}\\ \text{together}\\ \text{in 1 hour}\end{pmatrix}$$

Solution continued

	Hours to Do Job	Part of Job Done in 1 Hour
Danny	5	$\frac{1}{5}$
Mike	6	$\frac{1}{6}$
Together	t	$\frac{1}{t}$

Solution continued

From the table,

$$\frac{1}{5}+\frac{1}{6}=\frac{1}{t}$$ The model

$$\frac{6}{30}+\frac{5}{30}=\frac{1}{t}$$ LCD = 30 on the left.

$$\frac{11}{30}=\frac{1}{t}$$ Simplify.

$$11t=30$$ Multiply both sides by 30*t*.

$$t=\frac{30}{11}$$ Divide each side by 11.

Solution continued

Working together, Mike and Danny can do the job in 30/11 hours, or about 2 hours, 44 minutes.

They should make the golf date, because they will finish at 12:44 PM.