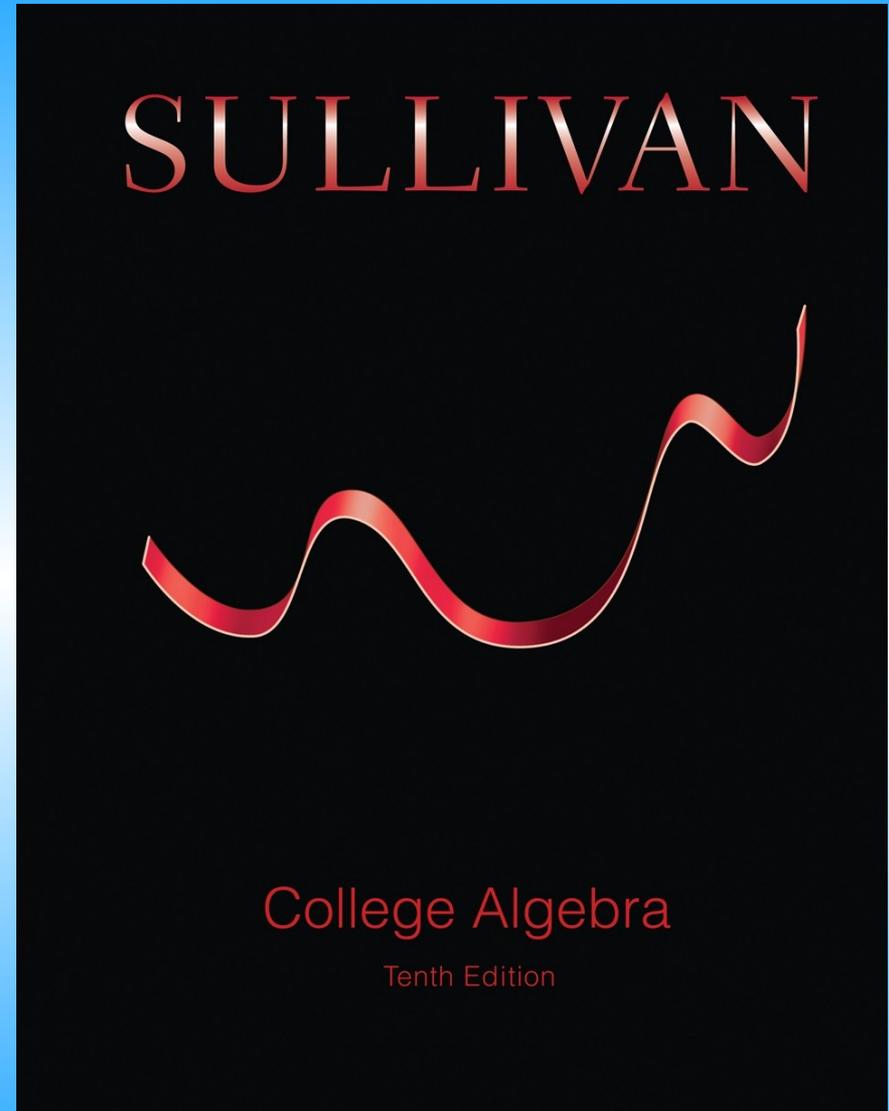


Chapter R

Section 5



R.5 Factoring Polynomials

- OBJECTIVES**
- 1** Factor the Difference of Two Squares and the Sum and Difference of Two Cubes (p. 50)
 - 2** Factor Perfect Squares (p. 51)
 - 3** Factor a Second-Degree Polynomial: $x^2 + Bx + C$ (p. 52)
 - 4** Factor by Grouping (p. 53)
 - 5** Factor a Second-Degree Polynomial: $Ax^2 + Bx + C$, $A \neq 1$ (p. 54)
 - 6** Complete the Square (p. 56)

Factor the Difference of Two Squares and the Sum and Difference of Two Cubes

Example

Factoring the Difference of Two Squares

Factor completely: $x^2 - 4$

Solution

Note that $x^2 - 4$ is the difference of two squares, x^2 and 2^2 .

$$x^2 - 4 = (x - 2)(x + 2)$$

Example

Factoring the Difference of Two Cubes

Factor completely: $x^3 - 1$

Solution

Because $x^3 - 1$ is the difference of two cubes, x^3 and 1^3 ,

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

Example

Factoring the Sum of Two Cubes

Factor completely: $x^3 + 8$

Solution

Because $x^3 + 8$ is the sum of two cubes, x^3 and 2^3 ,

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

Example

Factoring the Difference of Two Squares

Factor completely: $x^4 - 16$

Solution

Because $x^4 - 16$ is the difference of two squares, $x^4 = (x^2)^2$ and $16 = 4^2$,

$$x^4 - 16 = (x^2 - 4)(x^2 + 4)$$

But $x^2 - 4$ is also the difference of two squares. Then,

$$x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

Factor Perfect Squares

Example

Factoring a Perfect Square

Factor completely: $x^2 + 6x + 9$

Solution

The first term, x^2 , and the third term, $9 = 3^2$, are perfect squares. Because the middle term, $6x$, is twice the product of x and 3, we have a perfect square.

$$x^2 + 6x + 9 = (x + 3)^2$$

Example

Factoring a Perfect Square

Factor completely: $9x^2 - 6x + 1$

Solution

The first term, $9x^2 = (3x)^2$, and the third term, $1 = 1^2$, are perfect squares. Because the middle term, $-6x$, is -2 times the product of $3x$ and 1 , we have a perfect square.

$$9x^2 - 6x + 1 = (3x - 1)^2$$

Example

Factoring a Perfect Square

Factor completely: $25x^2 + 30x + 9$

Solution

The first term, $25x^2 = (5x)^2$, and the third term, $9 = 3^2$, are perfect squares. Because the middle term, $30x$, is twice the product of $5x$ and 3 , we have a perfect square.

$$25x^2 + 30x + 9 = (5x + 3)^2$$

Factor a Second-Degree Polynomial:

$$x^2 + Bx + C$$

To factor a second-degree polynomial $x^2 + Bx + C$, find integers whose product is C and whose sum is B . That is, if there are numbers a, b , where $ab = C$ and $a + b = B$, then

$$x^2 + Bx + C = (x + a)(x + b)$$

Example

Factoring a Trinomial

Factor completely: $x^2 - x - 12$

Solution

First determine all pairs of integers whose product is -12 , and then compute each sum.

Integers whose product is -12	1, -12	-1 , 12	2, -6	-2 , 6	3, -4	-3 , 4
Sum	-11	11	-4	4	-1	1

Since -1 is the coefficient of the middle term,

$$x^2 - x - 12 = (x + 3)(x - 4)$$

Example

Identifying a Prime Polynomial

Show that $x^2 + 9$ is prime.

Solution

First list the pairs of integers whose product is 9, and then compute their sums.

Integers whose product is 9	1, 9	-1, -9	3, 3	-3, -3
Sum	10	-10	6	-6

Since the coefficient of the middle term in $x^2 + 9 = x^2 + 0x + 9$ is 0 and none of the sums equals 0, we conclude that $x^2 + 9$ is prime.

Theorem

Any polynomial of the form $x^2 + a^2$, a real, is prime.

Factor by Grouping

Example

Factoring by Grouping

Factor completely by grouping: $3(x - 1)^2(x + 2)^4 + 4(x - 1)^3(x + 2)^3$

Solution

Here, $(x - 1)^2(x + 2)^3$ is a common factor of both $3(x - 1)^2(x + 2)^4$ and $4(x - 1)^3(x + 2)^3$. As a result,

$$\begin{aligned}3(x - 1)^2(x + 2)^4 + 4(x - 1)^3(x + 2)^3 &= (x - 1)^2(x + 2)^3[3(x + 2) + 4(x - 1)] \\ &= (x - 1)^2(x + 2)^3[3x + 6 + 4x - 4] \\ &= (x - 1)^2(x + 2)^3(7x + 2)\end{aligned}$$

Example

Factoring by Grouping

Factor completely by grouping: $x^3 - 4x^2 + 2x - 8$

Solution

To see whether factoring by grouping will work, group the first two terms and the last two terms. Then look for a common factor in each group. In this example, factor x^2 from $x^3 - 4x^2$ and 2 from $2x - 8$. The remaining factor in each case is the same, $x - 4$. This means that factoring by grouping will work, as follows:

$$\begin{aligned}x^3 - 4x^2 + 2x - 8 &= (x^3 - 4x^2) + (2x - 8) \\ &= x^2(x - 4) + 2(x - 4) \\ &= (x - 4)(x^2 + 2)\end{aligned}$$

Since $x^2 + 2$ and $x - 4$ are prime, the factorization is complete.

Factoring a Second-Degree Polynomial: $Ax^2+Bx+C, A \neq 1$

Steps for Factoring $Ax^2 + Bx + C$, When $A \neq 1$ and $A, B,$ and C Have No Common Factors

STEP 1: Find the value of AC .

STEP 2: Find a pair of integers whose product is AC and that add up to B . That is, find a and b such that $ab = AC$ and $a + b = B$.

STEP 3: Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.

STEP 4: Factor this last expression by grouping.

Example

Factoring a Trinomial

Factor completely: $2x^2 - x - 6$

Solution

Comparing $2x^2 - x - 6$ to $Ax^2 + Bx + C$, we find that $A = 2$, $B = -1$, and $C = -6$.

STEP 1: The value of AC is $2 \cdot (-6) = -12$.

STEP 2: Determine the pairs of integers whose product is $AC = -12$ and compute their sums.

Integers whose product is -12	1, -12	-1, 12	2, -6	-2, 6	3, -4	-3, 4
Sum	-11	11	-4	4	-1	1

STEP 3: The integers whose product is -12 that add up to $B = -1$ are -4 and 3 .

$$2x^2 - x - 6 = 2x^2 - 4x + 3x - 6$$

STEP 4: Factor by grouping.

$$\begin{aligned}2x^2 - 4x + 3x - 6 &= (2x^2 - 4x) + (3x - 6) \\ &= 2x(x - 2) + 3(x - 2) \\ &= (x - 2)(2x + 3)\end{aligned}$$

As a result,

$$2x^2 - x - 6 = (x - 2)(2x + 3)$$

Complete the Square

Completing the Square of $x^2 + bx$

Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of b in $x^2 + bx$ and compute $\left(\frac{1}{2}b\right)^2$.

Example

Completing the Square

Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2} \cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y + 4)^2$
$x^2 + 12x$	$\left(\frac{1}{2} \cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x + 6)^2$
$a^2 - 20a$	$\left(\frac{1}{2} \cdot (-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a - 10)^2$
$p^2 - 5p$	$\left(\frac{1}{2} \cdot (-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

Figure: Completing the Square

