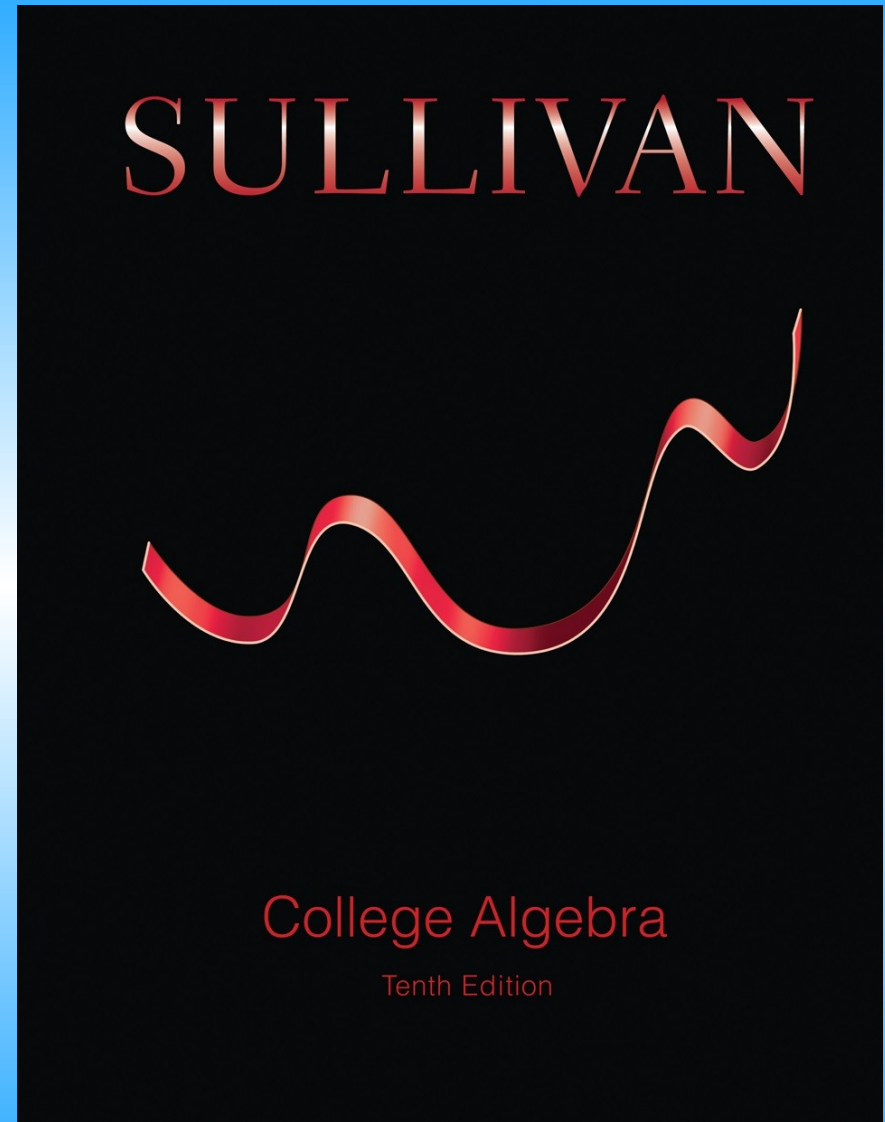


Chapter R

Section 4



R.4 Polynomials

- OBJECTIVES**
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 - 7** Work with Polynomials in Two Variables (p. 47)

Recognize Monomials

Definition

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial.

Recognize Polynomials

Definition

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (1)$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, $a_n x^n$ is called the **leading term**, and n is the **degree** of the polynomial.

Example

Examples of Polynomials

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

Add and Subtract Polynomials

Example

Adding Polynomials

Find the sum of the polynomials:

$$8x^3 - 2x^2 + 6x - 2 \quad \text{and} \quad 3x^4 - 2x^3 + x^2 + x$$

Solution

We shall find the sum in two ways.

Horizontal Addition: The idea here is to group the like terms and then combine them.

$$\begin{aligned} & (8x^3 - 2x^2 + 6x - 2) + (3x^4 - 2x^3 + x^2 + x) \\ &= 3x^4 + (8x^3 - 2x^3) + (-2x^2 + x^2) + (6x + x) - 2 \\ &= 3x^4 + 6x^3 - x^2 + 7x - 2 \end{aligned}$$

Vertical Addition: The idea here is to vertically line up the like terms in each polynomial and then add the coefficients.

$$\begin{array}{rcccccc} & x^4 & & x^3 & & x^2 & & x^1 & & x^0 \\ & & & 8x^3 & - & 2x^2 & + & 6x & - & 2 \\ + & 3x^4 & - & 2x^3 & + & x^2 & + & x & & \\ \hline & 3x^4 & + & 6x^3 & - & x^2 & + & 7x & - & 2 \end{array}$$

Example

Subtracting Polynomials

Find the difference: $(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$

Solution

Horizontal Subtraction:

$$(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$$

$$= 3x^4 - 4x^3 + 6x^2 - 1 + \underbrace{(-2x^4 + 8x^2 + 6x - 5)}$$

Be sure to change the sign of each term in the second polynomial.

$$= (3x^4 - 2x^4) + (-4x^3) + (6x^2 + 8x^2) + 6x + (-1 - 5)$$

↑
Group like terms.

$$= x^4 - 4x^3 + 14x^2 + 6x - 6$$

Solution continued

Vertical Subtraction: We line up like terms, change the sign of each coefficient of the second polynomial, and add.

$$\begin{array}{r} \begin{array}{cccccc} x^4 & x^3 & x^2 & x^1 & x^0 & \\ 3x^4 & -4x^3 & +6x^2 & & -1 & \\ - [2x^4 & & -8x^2 & -6x & +5] & \end{array} = + \begin{array}{r} \begin{array}{cccccc} x^4 & x^3 & x^2 & x^1 & x^0 & \\ 3x^4 & -4x^3 & +6x^2 & & -1 & \\ -2x^4 & & +8x^2 & +6x & -5 & \\ \hline x^4 & -4x^3 & +14x^2 & +6x & -6 & \end{array} \end{array}$$

Multiply Polynomials

Example

Multiplying Polynomials

Find the product: $(2x + 5)(x^2 - x + 2)$

Solution

Horizontal Multiplication:

$$\begin{aligned}(2x + 5)(x^2 - x + 2) &= 2x(x^2 - x + 2) + 5(x^2 - x + 2) \\ &\quad \uparrow \\ &\quad \text{Distributive Property} \\ &= (2x \cdot x^2 - 2x \cdot x + 2x \cdot 2) + (5 \cdot x^2 - 5 \cdot x + 5 \cdot 2) \\ &\quad \uparrow \\ &\quad \text{Distributive Property} \\ &= (2x^3 - 2x^2 + 4x) + (5x^2 - 5x + 10) \\ &\quad \uparrow \\ &\quad \text{Law of Exponents} \\ &= 2x^3 + 3x^2 - x + 10 \\ &\quad \uparrow \\ &\quad \text{Combine like terms.}\end{aligned}$$

Solution continued

Vertical Multiplication: The idea here is very much like multiplying a two-digit number by a three-digit number.

$$\begin{array}{r} x^2 - x + 2 \\ 2x + 5 \\ \hline 2x^3 - 2x^2 + 4x \\ (+) \quad 5x^2 - 5x + 10 \\ \hline 2x^3 + 3x^2 - x + 10 \end{array}$$

This line is $2x(x^2 - x + 2)$.
This line is $5(x^2 - x + 2)$.
Sum of the above two lines

Know Formulas for Special Products

Example

Using FOIL

$$(a) \quad (x - 3)(x + 3) = x^2 + 3x - 3x - 9 = x^2 - 9$$

F O I L

$$(b) \quad (x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$$

$$(c) \quad (x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$$

$$(d) \quad (x + 3)(x + 1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$$

$$(e) \quad (2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$$

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \quad (2)$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \quad (3a)$$

$$(x - a)^2 = x^2 - 2ax + a^2 \quad (3b)$$

Example

Using Special Product Formulas

(a) $(x - 5)(x + 5) = x^2 - 5^2 = x^2 - 25$

Difference of two squares

(b) $(x + 7)^2 = x^2 + 2 \cdot 7 \cdot x + 7^2 = x^2 + 14x + 49$

Square of a binomial

(c) $(2x + 1)^2 = (2x)^2 + 2 \cdot 1 \cdot 2x + 1^2 = 4x^2 + 4x + 1$

Notice that we used $2x$ in place of x in formula (3a).

(d) $(3x - 4)^2 = (3x)^2 - 2 \cdot 4 \cdot 3x + 4^2 = 9x^2 - 24x + 16$

Replace x by $3x$ in formula (3b).

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \quad (4a)$$

$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \quad (4b)$$

Example

Forming the Difference of Two Cubes

$$\begin{aligned}(x - 1)(x^2 + x + 1) &= x(x^2 + x + 1) - 1(x^2 + x + 1) \\ &= x^3 + x^2 + x - x^2 - x - 1 \\ &= x^3 - 1\end{aligned}$$

Example

Forming the Sum of Two Cubes

$$\begin{aligned}(x + 2)(x^2 - 2x + 4) &= x(x^2 - 2x + 4) + 2(x^2 - 2x + 4) \\&= x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 \\&= x^3 + 8\end{aligned}$$

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \quad (5)$$

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \quad (6)$$

Divide Polynomials Using Long Division

$$(\text{Quotient}) (\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

Example

Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \text{ is divided by } x^2 + 1$$

Solution

Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

STEP 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2 . Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{r} 3x \\ x^2 + 1 \overline{) 3x^3 + 4x^2 + x + 7} \end{array}$$

STEP 2: Multiply $3x$ by $x^2 + 1$, and enter the result below the dividend.

$$\begin{array}{r} 3x \\ x^2 + 1 \overline{) 3x^3 + 4x^2 + x + 7} \\ \underline{3x^3 \qquad + 3x} \end{array} \quad \leftarrow 3x \cdot (x^2 + 1) = 3x^3 + 3x$$

$\uparrow$
 Align the $3x$ term under the x
 to make the next step easier.

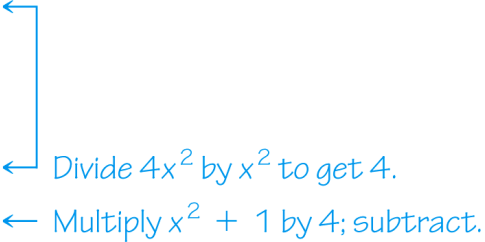
STEP 3: Subtract and bring down the remaining terms.

$$\begin{array}{r} 3x \\ x^2 + 1 \overline{) 3x^3 + 4x^2 + x + 7} \\ \underline{3x^3 \qquad + 3x} \\ 4x^2 - 2x + 7 \end{array} \quad \begin{array}{l} \leftarrow \text{Subtract (change the signs and add).} \\ \leftarrow \text{Bring down the } 4x^2 \text{ and the } 7. \end{array}$$

Solution continued

STEP 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$\begin{array}{r}
 3x + 4 \\
 x^2 + 1 \overline{) 3x^3 + 4x^2 + x + 7} \\
 \underline{3x^3 + 3x} \\
 4x^2 - 2x + 7 \\
 \underline{4x^2 + 4} \\
 -2x + 3
 \end{array}$$



Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

✓ **Check:** (Quotient) (Divisor) + Remainder

$$\begin{aligned}
 &= (3x + 4)(x^2 + 1) + (-2x + 3) \\
 &= 3x^3 + 3x + 4x^2 + 4 + (-2x + 3) \\
 &= 3x^3 + 4x^2 + x + 7 = \text{Dividend}
 \end{aligned}$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

Theorem

Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q . The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q .

Work with Polynomials in Two Variables

Example

Using a Special Product Formula

To multiply $(2x - y)^2$, use the Squares of Binomials formula (3b) with $2x$ instead of x and with y instead of a .

$$\begin{aligned}(2x - y)^2 &= (2x)^2 - 2 \cdot y \cdot 2x + y^2 \\ &= 4x^2 - 4xy + y^2\end{aligned}$$