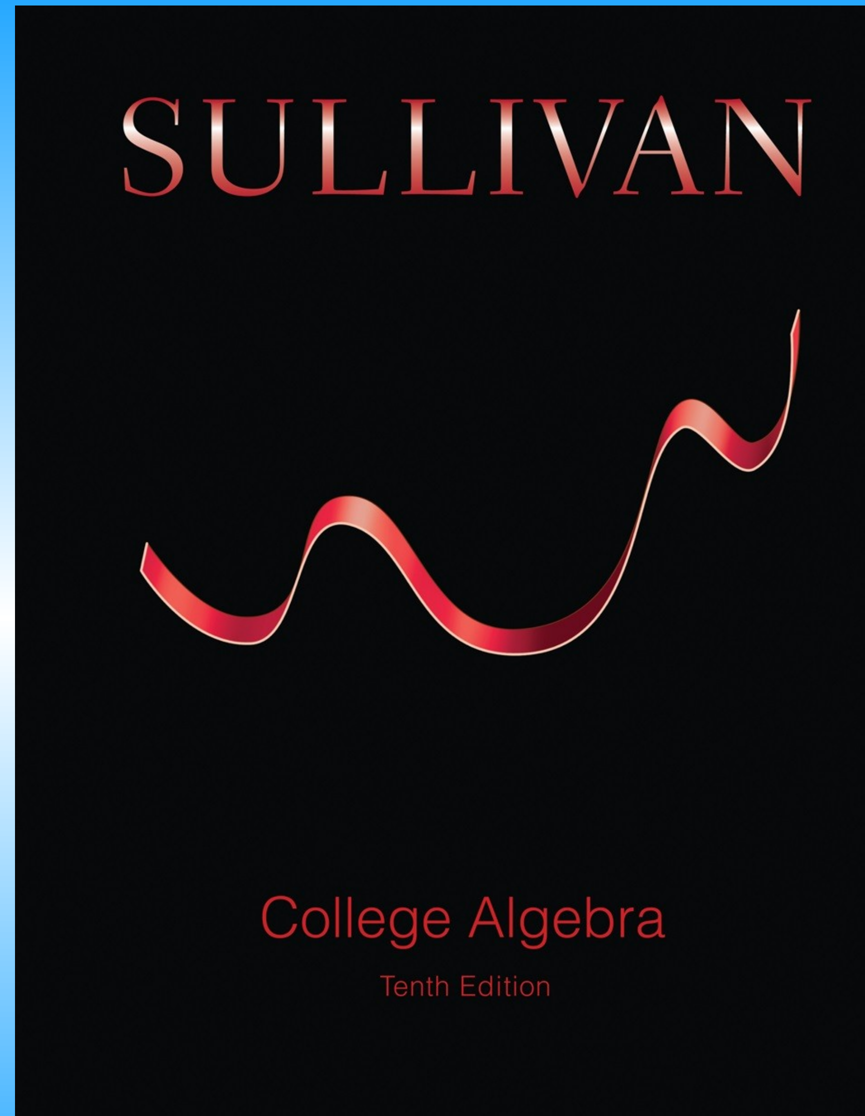
Chapter 1

Section 2



1.2 Quadratic Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Factoring Polynomials (Section R.5, pp. 49–55)
- Zero-Product Property (Section R.1, p. 13)
- Square Roots (Section R.2, pp. 23–24)
- Complete the Square (Section R.5, p. 56)

Now Work the **'Are You Prepared?'** problems on page 101.

OBJECTIVES

1. Solve a Quadratic Equation by Factoring (p. 93)
2. Solve a Quadratic Equation by Completing the Square (p. 95)
3. Solve a Quadratic Equation Using the Quadratic Formula (p. 96)
4. Solve Problems That Can Be Modeled by Quadratic Equations (p. 99)

Definition

A **quadratic equation** is an equation equivalent to one of the form

$$ax^2 + bx + c = 0 \qquad \textbf{(1)}$$

where a, b, and c are real numbers and $a \neq 0$.

Solve a Quadratic Equation by Factoring

Example

Solving a Quadratic Equation by Factoring

Solve the equations:

(a) $x^2 + 6x = 0$

(b) $2x^2 = x + 3$

Solution

(a) The equation is in the standard form specified in equation (1). The left side may be factored as

$$x^2 + 6x = 0$$

$$x(x + 6) = 0 \quad \text{Factor.}$$

Using the Zero-Product Property, set each factor equal to 0 and then solve the resulting first-degree equations.

$$x = 0 \quad \text{or} \quad x + 6 = 0 \qquad \text{Zero-Product Property}$$

$$x = 0 \quad \text{or} \qquad x = -6 \qquad \text{Solve.}$$

The solution set is $\{0, -6\}$.

Solution continued

(b) Place the equation $2x^2 = x + 3$ in standard form by subtracting x and 3 from both sides.

$$2x^2 = x + 3$$

$$2x^2 - x - 3 = 0 \qquad \text{Subtract } x \text{ and 3 from both sides.}$$

The left side may now be factored as

$$(2x - 3)(x + 1) = 0 \qquad \text{Factor.}$$

so that

$$2x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{Zero-Product Property}$$

$$x = \frac{3}{2} \qquad\qquad x = -1 \qquad \text{Solve.}$$

The solution set is $\left\{-1, \frac{3}{2}\right\}$.

The Square Root Method

If $x^2 = p$ and $p \geq 0$, then $x = \sqrt{p}$ or $x = -\sqrt{p}$. **(3)**

Example

Solving a Quadratic Equation Using the Square Root Method

Solve each equation.

(a) $x^2 = 5$ (b) $(x - 2)^2 = 16$

Solution

(a) Use the Square Root Method to get

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$ Use the Square Root Method.

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5}$$

The solution set is $\{-\sqrt{5}, \sqrt{5}\}$.

(b) Use the Square Root Method to get

$$(x-2)^2 = 16$$

$$x - 2 = \pm\sqrt{16}$$ Use the Square Root Method.

$$x - 2 = \pm 4$$ $\sqrt{16} = 4$

$$x - 2 = 4 \quad \text{or} \quad x - 2 = -4$$

$$x = 6 \quad \text{or} \quad x = -2$$

The solution set is $\{-2, 6\}$.

Solve a Quadratic Equation by Completing the Square

Example

Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $x^2 + 5x + 4 = 0$

Solution

Always begin this procedure by rearranging the equation so that the constant is on the right side.

$$x^2 + 5x + 4 = 0$$
$$x^2 + 5x = -4$$

Since the coefficient of x^2 is 1, complete the square on the left side by adding $\left(\frac{1}{2}\cdot 5\right)^2 = \frac{25}{4}$. Of course, in an equation, whatever is added to the left side must also be added to the right side. So add $\frac{25}{4}$ to *both* sides.

$$x^2 + 5x + \frac{25}{4} = -4 + \frac{25}{4}$$ Add $\frac{25}{4}$ to both sides.

$$\left(x + \frac{5}{2}\right)^2 = \frac{9}{4}$$ Factor the left side.

$$x + \frac{5}{2} = \pm\sqrt{\frac{9}{4}}$$ Use the Square Root Method.

$$x + \frac{5}{2} = \pm\frac{3}{2}$$ $\sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

$$x = -\frac{5}{2} \pm \frac{3}{2}$$

$$x = -\frac{5}{2} + \frac{3}{2} = -1 \quad \text{or} \quad x = -\frac{5}{2} - \frac{3}{2} = -4$$

The solution set is $\{-4, -1\}$.

Example

Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $2x^2 - 8x - 5 = 0$

Solution

First, rewrite the equation so that the constant is on the right side.

$$2x^2 - 8x - 5 = 0$$

$$2x^2 - 8x = 5 \quad \text{Add 5 to both sides.}$$

Solution continued

Next, divide both sides by 2 so that the coefficient of x^2 is 1. (This enables us to complete the square at the next step.)

$$x^2 - 4x = \frac{5}{2}$$

Finally, complete the square by adding 4 to both sides.

$$x^2 - 4x + 4 = \frac{5}{2} + 4 \qquad \text{Add 4 to both sides.}$$

$$(x-2)^2 = \frac{13}{2} \qquad \text{Factor on the left; simplify on the right.}$$

$$x - 2 = \pm\sqrt{\frac{13}{2}} \qquad \text{Use the Square Root Method.}$$

$$x - 2 = \pm\frac{\sqrt{26}}{2} \qquad \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}$$

$$x = 2 \pm \frac{\sqrt{26}}{2}$$

The solution set is $\left\{2 - \frac{\sqrt{26}}{2}, 2 + \frac{\sqrt{26}}{2}\right\}$.

Solve a Quadratic Equation Using the Quadratic Formula

Theorem

Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

If $b^2 - 4ac < 0$, this equation has no real solution.
If $b^2 - 4ac \geq 0$, the real solution(s) of this equation is (are) given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(5)}$$

The Discriminant

Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$:

1. If $b^2 - 4ac > 0$, there are two unequal real solutions.
2. If $b^2 - 4ac = 0$, there is a repeated solution, a double root.
3. If $b^2 - 4ac < 0$, there is no real solution.

Example

Solving a Quadratic Equation Using the Quadratic Formula

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 - 5x + 1 = 0$$

Solution

The equation is in standard form, so compare it to $ax^2 + bx + c = 0$ to find a, b, and c.

$$3x^2 - 5x + 1 = 0$$
$$ax^2 + bx + c = 0 \quad a = 3, b = -5, c = 1$$

With $a = 3$, $b = -5$, and $c = 1$, evaluate the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$$

Since $b^2 - 4ac > 0$, there are two real solutions, which can be found using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{13}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\frac{5 - \sqrt{13}}{6}, \frac{5 + \sqrt{13}}{6}\right\}$.

Example

Solving a Quadratic Equation Using the Quadratic Formula

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 + 2 = 4x$$

Solution

The equation, as given, is not in standard form.

$$3x^2 + 2 = 4x$$

$$3x^2 - 4x + 2 = 0 \quad \text{Put in standard form.}$$

$$ax^2 + bx + c = 0 \quad \text{Compare to standard form.}$$

With $a = 3$, $b = -4$, and $c = 2$, the discriminant is

$$b^2 - 4ac = (-4)^2 - 4(3)(2) = 16 - 24 = -8$$

Since $b^2 - 4ac < 0$, the equation has no real solution.

Example

Solving a Quadratic Equation Using the Quadratic Formula

Find the real solutions, if any, of the equation: $9 + \frac{3}{x} - \frac{2}{x^2} = 0, \ x \neq 0$

Solution

In its present form, the equation

$$9 + \frac{3}{x} - \frac{2}{x^2} = 0$$

is not a quadratic equation. However, it can be transformed into one by multiplying each side by x^2. The result is

$$9x^2 + 3x - 2 = 0$$

Although we multiplied each side by x^2, we know that $x^2 \neq 0$ (do you see why?), so this quadratic equation is equivalent to the original equation.

Using $a = 9$, $b = 3$, and $c = -2$, the discriminant is

$$b^2 - 4ac = 3^2 - 4(9)(-2) = 9 + 72 = 81$$

Solution

Since $b^2 - 4ac > 0$, the new equation has two real solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{81}}{2(9)} = \frac{-3 \pm 9}{18}$$

$$x = \frac{-3 + 9}{18} = \frac{6}{18} = \frac{1}{3} \quad \text{or} \quad x = \frac{-3 - 9}{18} = \frac{-12}{18} = -\frac{2}{3}$$

The solution set is $\left\{-\frac{2}{3}, \frac{1}{3}\right\}$.

Solve Problems That Can Be Modeled by Quadratic Equations

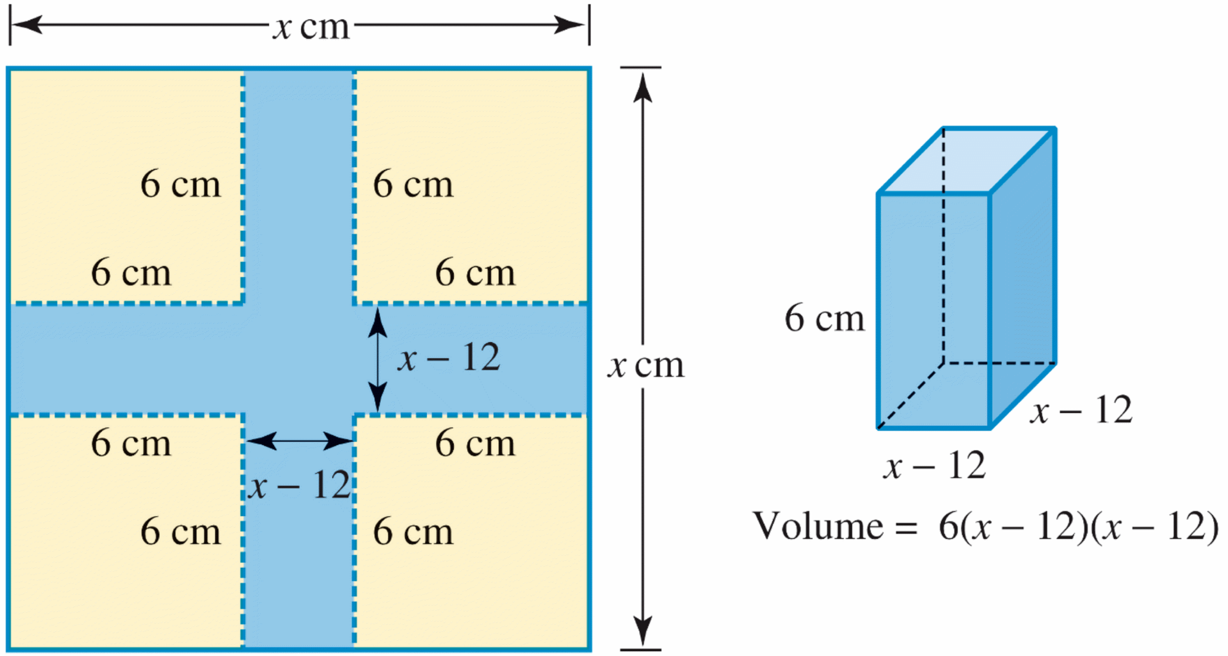
Example

Constructing a Box

From each corner of a square piece of sheet metal, remove a square of side 6 centimeters. Turn up the edges to form an open box. If the box is to hold 486 cubic centimeters (cm^3), what should be the dimensions of the piece of sheet metal?

Solution

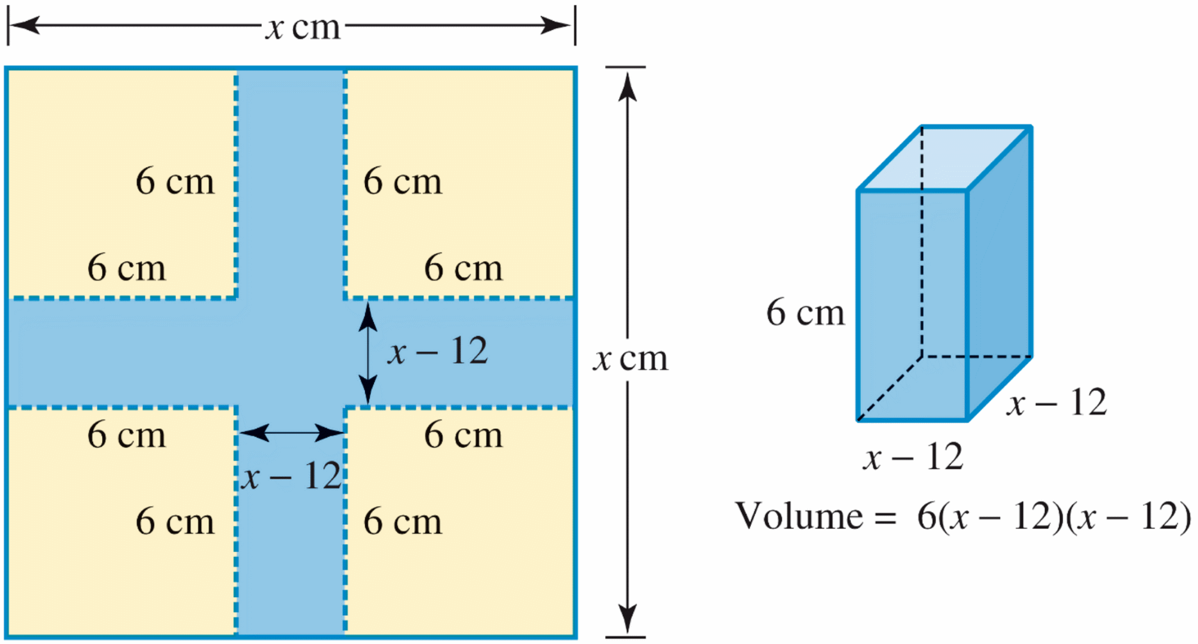
Use the figure as a guide. We have labeled by x the length of a side of the square piece of sheet metal. The box will be of height 6 centimeters, and its square base will measure $x – 12$ on each side.



Solution continued

The volume V (Length × Width × Height) of the box is therefore

$$V = (x-12)(x-12)\cdot 6 = 6(x-12)^2$$



Solution continued

Since the volume of the box is to be 486 cm^3, we have

$$
\begin{aligned}
6(x-12)^2 &= 486 \\
(x-12)^2 &= 81 \\
x-12 &= \pm 9 \\
x &= 21 \quad \text{or} \quad x = 3
\end{aligned}
$$

Discard the solution $x = 3$ (do you see why?) and conclude that the sheet metal should be 21 cm by 21 cm.