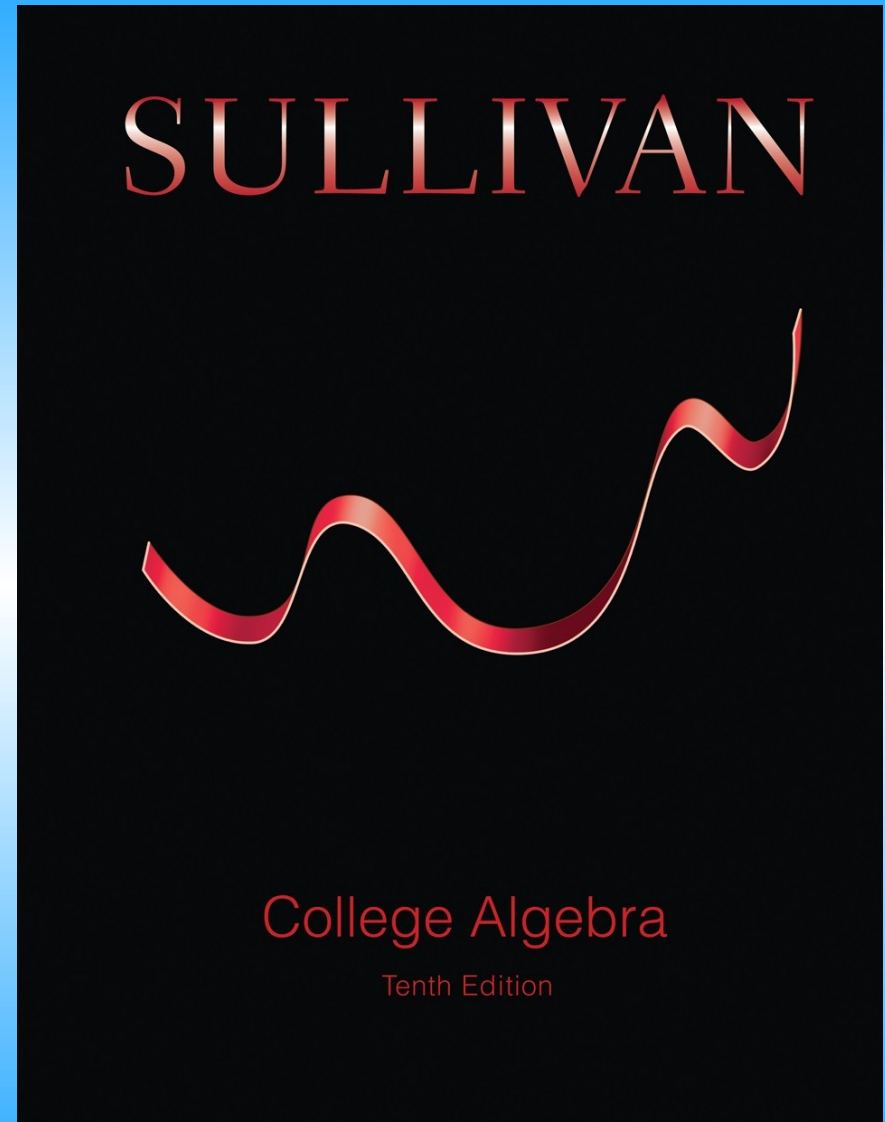


Chapter R

Section 8



R.8 n th Roots; Rational Exponents

PREPARING FOR THIS SECTION Before getting started, review the following:

- Exponents, Square Roots (Section R.2, pp. 21–24)



Now Work the 'Are You Prepared?' problems on page 78.

- OBJECTIVES**
- 1** Work with n th Roots (p. 73)
 - 2** Simplify Radicals (p. 74)
 - 3** Rationalize Denominators (p. 75)
 - 4** Simplify Expressions with Rational Exponents (p. 76)

Work with n th Roots

Definition

The **principal n th root of a real number a** , $n \geq 2$ an integer, symbolized by $\sqrt[n]{a}$, is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad a = b^n$$

where $a \geq 0$ and $b \geq 0$ if n is even and a, b are any real numbers if n is odd.

Example 1

Simplifying Principal n th Roots

$$(a) \sqrt[3]{8} = \sqrt[3]{2^3} = 2$$

$$(c) \sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$$

$$(b) \sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$$

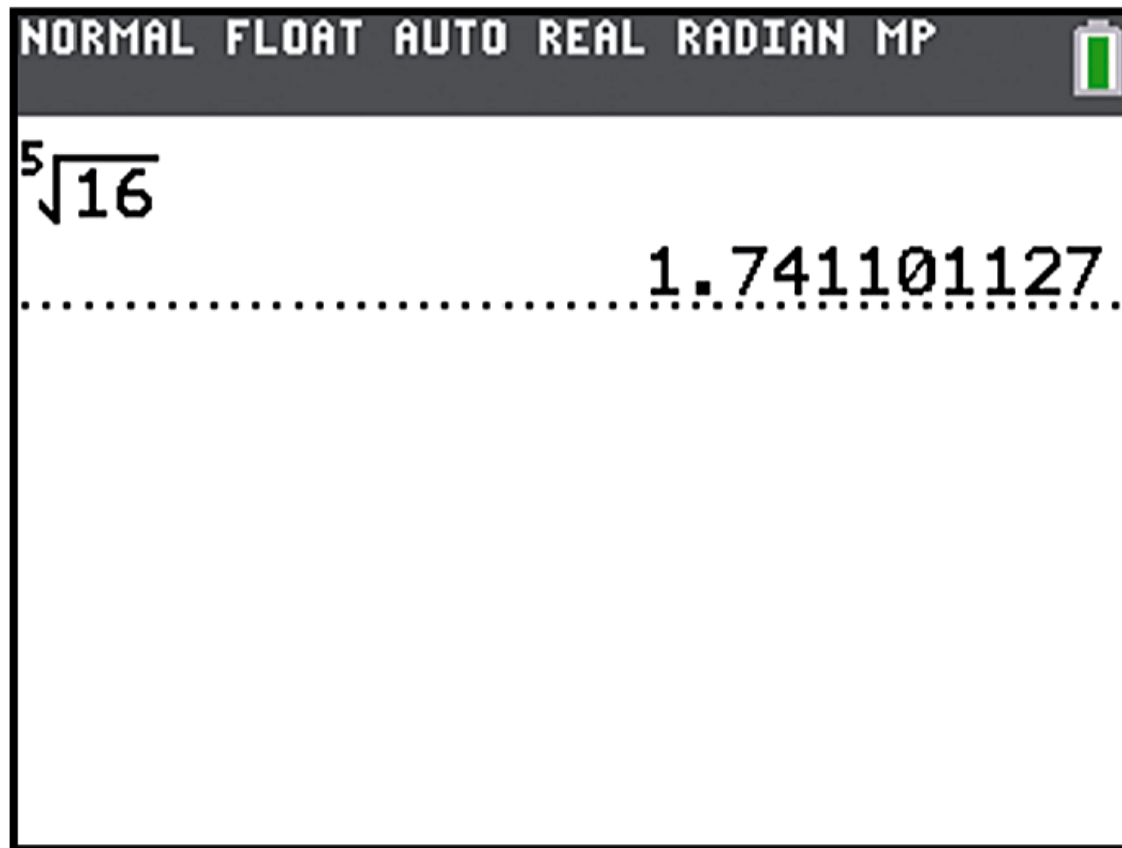
$$(d) \sqrt[6]{(-2)^6} = |-2| = 2$$

In general, if $n \geq 2$ is an integer and a is a real number, we have

$$\sqrt[n]{a^n} = a \quad \text{if } n \geq 3 \text{ is odd} \quad \mathbf{(1a)}$$

$$\sqrt[n]{a^n} = |a| \quad \text{if } n \geq 2 \text{ is even} \quad \mathbf{(1b)}$$

Figure



Simplify Radicals

Properties of Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b} \quad (2a)$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad b \neq 0 \quad (2b)$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (2c)$$

Example

Simplifying Radicals

$$(a) \sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$$

$\uparrow$ Factor out 16, a perfect square. $\uparrow$ (2a)

$$(b) \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = \sqrt[3]{2^3} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$$

$\uparrow$ Factor out 8, a perfect cube. $\uparrow$ (2a)

$$(c) \sqrt[3]{-16x^4} = \sqrt[3]{-8 \cdot 2 \cdot x^3 \cdot x} = \sqrt[3]{(-8x^3)(2x)}$$

$\uparrow$ Factor perfect cubes inside radical. $\uparrow$ Group perfect cubes.

$$= \sqrt[3]{(-2x)^3 \cdot 2x} = \sqrt[3]{(-2x)^3} \cdot \sqrt[3]{2x} = -2x\sqrt[3]{2x}$$

$\uparrow$ (2a)

$$(d) \sqrt[4]{\frac{16x^5}{81}} = \sqrt[4]{\frac{2^4 x^4 x}{3^4}} = \sqrt[4]{\left(\frac{2x}{3}\right)^4 \cdot x} = \sqrt[4]{\left(\frac{2x}{3}\right)^4} \cdot \sqrt[4]{x} = \left|\frac{2x}{3}\right| \sqrt[4]{x}$$

Example

Combining Like Radicals

$$\begin{aligned}\text{(a)} \quad -8\sqrt{12} + \sqrt{3} &= -8\sqrt{4 \cdot 3} + \sqrt{3} \\ &= -8 \cdot \sqrt{4} \sqrt{3} + \sqrt{3} \\ &= -16\sqrt{3} + \sqrt{3} = -15\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \sqrt[3]{8x^4} + \sqrt[3]{-x} + 4\sqrt[3]{27x} &= \sqrt[3]{2^3x^3x} + \sqrt[3]{-1 \cdot x} + 4\sqrt[3]{3^3x} \\ &= \sqrt[3]{(2x)^3} \cdot \sqrt[3]{x} + \sqrt[3]{-1} \cdot \sqrt[3]{x} + 4\sqrt[3]{3^3} \cdot \sqrt[3]{x} \\ &= 2x\sqrt[3]{x} - 1 \cdot \sqrt[3]{x} + 12\sqrt[3]{x} \\ &= (2x + 11)\sqrt[3]{x}\end{aligned}$$

Rationalize Denominators

**If a Denominator
Contains the Factor**

$$\sqrt{3}$$

$$\sqrt{3} + 1$$

$$\sqrt{2} - 3$$

$$\sqrt{5} - \sqrt{3}$$

$$\sqrt[3]{4}$$

Multiply by

$$\sqrt{3}$$

$$\sqrt{3} - 1$$

$$\sqrt{2} + 3$$

$$\sqrt{5} + \sqrt{3}$$

$$\sqrt[3]{2}$$

**To Obtain a Denominator
Free of Radicals**

$$(\sqrt{3})^2 = 3$$

$$(\sqrt{3})^2 - 1^2 = 3 - 1 = 2$$

$$(\sqrt{2})^2 - 3^2 = 2 - 9 = -7$$

$$(\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$$

$$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$$

Example

Rationalizing Denominators

Rationalize the denominator of each expression:

(a) $\frac{1}{\sqrt{3}}$

(b) $\frac{5}{4\sqrt{2}}$

(c) $\frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}}$

Solution

- (a) The denominator contains the factor $\sqrt{3}$, so we multiply the numerator and denominator by $\sqrt{3}$ to obtain

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{(\sqrt{3})^2} = \frac{\sqrt{3}}{3}$$

- (b) The denominator contains the factor $\sqrt{2}$, so we multiply the numerator and denominator by $\sqrt{2}$ to obtain

$$\frac{5}{4\sqrt{2}} = \frac{5}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{4(\sqrt{2})^2} = \frac{5\sqrt{2}}{4 \cdot 2} = \frac{5\sqrt{2}}{8}$$

- (c) The denominator contains the factor $\sqrt{3} - 3\sqrt{2}$, so we multiply the numerator and denominator by $\sqrt{3} + 3\sqrt{2}$ to obtain

$$\begin{aligned} \frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}} &= \frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}} \cdot \frac{\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + 3\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3} + 3\sqrt{2})}{(\sqrt{3})^2 - (3\sqrt{2})^2} \\ &= \frac{\sqrt{2}\sqrt{3} + 3(\sqrt{2})^2}{3 - 18} = \frac{\sqrt{6} + 6}{-15} = -\frac{6 + \sqrt{6}}{15} \end{aligned}$$

Simplify Expressions with Rational Exponents

Definition

If a is a real number and $n \geq 2$ is an integer, then

$$a^{1/n} = \sqrt[n]{a} \quad (3)$$

provided that $\sqrt[n]{a}$ exists.

Example

Writing Expressions Containing Fractional Exponents as Radicals

$$(a) \ 4^{1/2} = \sqrt{4} = 2$$

$$(b) \ 8^{1/2} = \sqrt{8} = 2\sqrt{2}$$

$$(c) \ (-27)^{1/3} = \sqrt[3]{-27} = -3$$

$$(d) \ 16^{1/3} = \sqrt[3]{16} = 2\sqrt[3]{2}$$

Definition

If a is a real number and m and n are integers containing no common factors, with $n \geq 2$, then

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (4)$$

provided that $\sqrt[n]{a}$ exists.

Example

Using Equation (4)

$$(a) \quad 4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$$

$$(b) \quad (-8)^{4/3} = (\sqrt[3]{-8})^4 = (-2)^4 = 16$$

$$(c) \quad (32)^{-2/5} = (\sqrt[5]{32})^{-2} = 2^{-2} = \frac{1}{4}$$

$$(d) \quad 25^{6/4} = 25^{3/2} = (\sqrt{25})^3 = 5^3 = 125$$

Example

Simplifying Expressions Containing Rational Exponents

Simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

(a) $(x^{2/3}y)(x^{-2}y)^{1/2}$

(b) $\left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3}$

(c) $\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2}$

Solution

$$\begin{aligned} \text{(a)} \quad (x^{2/3}y)(x^{-2}y)^{1/2} &= (x^{2/3}y) [(x^{-2})^{1/2}y^{1/2}] \\ &= x^{2/3}yx^{-1}y^{1/2} \\ &= (x^{2/3} \cdot x^{-1})(y \cdot y^{1/2}) \\ &= x^{-1/3}y^{3/2} \\ &= \frac{y^{3/2}}{x^{1/3}} \end{aligned}$$

$$\text{(b)} \quad \left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3} = \left(\frac{y^{2/3}}{2x^{1/3}}\right)^3 = \frac{(y^{2/3})^3}{(2x^{1/3})^3} = \frac{y^2}{2^3(x^{1/3})^3} = \frac{y^2}{8x}$$

$$\text{(c)} \quad \left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2} = \left(\frac{9x^{2-(1/3)}}{y^{1-(1/3)}}\right)^{1/2} = \left(\frac{9x^{5/3}}{y^{2/3}}\right)^{1/2} = \frac{9^{1/2}(x^{5/3})^{1/2}}{(y^{2/3})^{1/2}} = \frac{3x^{5/6}}{y^{1/3}}$$

Example

Factoring an Expression Containing Rational Exponents

Factor and simplify: $\frac{4}{3}x^{1/3}(2x + 1) + 2x^{4/3}$

Solution

Begin by writing $2x^{4/3}$ as a fraction with 3 as the denominator.

$$\frac{4}{3}x^{1/3}(2x + 1) + 2x^{4/3} = \frac{4x^{1/3}(2x + 1)}{3} + \frac{6x^{4/3}}{3} = \frac{4x^{1/3}(2x + 1) + 6x^{4/3}}{3}$$

↑
Add the two fractions

$$= \frac{2x^{1/3}[2(2x + 1) + 3x]}{3} = \frac{2x^{1/3}(7x + 2)}{3}$$

↑
2 and $x^{1/3}$ are common factors

↑
Simplify