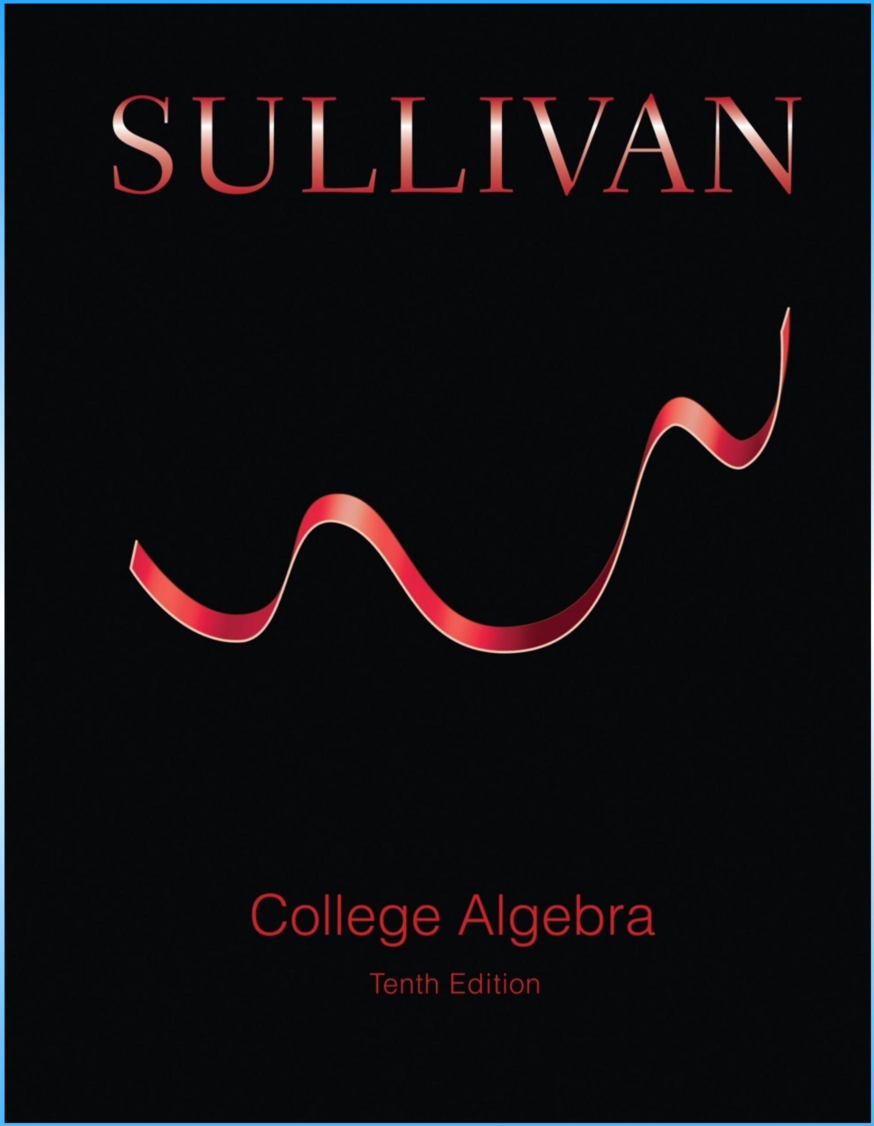
Chapter 6

Section 5



6.5 Properties of Logarithms

OBJECTIVES

1. Work with the Properties of Logarithms (p. 452)
2. Write a Logarithmic Expression as a Sum or Difference of Logarithms (p. 454)
3. Write a Logarithmic Expression as a Single Logarithm (p. 455)
4. Evaluate Logarithms Whose Base Is Neither 10 Nor *e* (p. 457)

Work with the Properties of Logarithms

$$\log_a 1 = 0 \qquad \log_a a = 1$$

Theorem

Properties of Logarithms

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad \textbf{(1)}$$

The logarithm with base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad \textbf{(2)}$$

Example

Using Properties (1) and (2)

(a) $2^{\log_2 \pi} = \pi$ (b) $\log_{0.2} 0.2^{-\sqrt{2}} = -\sqrt{2}$ (c) $\ln e^{kt} = kt$

Theorem

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \qquad \textbf{(3)}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad \textbf{(4)}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \qquad \textbf{(5)}$$

$$a^r = e^{r \ln a} \qquad \textbf{(6)}$$

Write a Logarithmic Expression as a Sum or Difference of Logarithms

Example

Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log_a\left(x\sqrt{x^2 + 1}\right)$, $x > 0$, as a sum of logarithms. Express all powers as factors.

Solution

$$\log_a\left(x\sqrt{x^2+1}\right) = \log_a x + \log_a \sqrt{x^2+1} \qquad \log_a(M \cdot N) = \log_a M + \log_a N$$

$$= \log_a x + \log_a (x^2+1)^{1/2}$$

$$= \log_a x + \frac{1}{2}\log_a(x^2+1) \qquad \log_a M^r = r\log_a M$$

Example

Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln \frac{x^2}{(x-1)^3} \qquad x > 1$$

as a difference of logarithms. Express all powers as factors.

Solution

$$\ln \frac{x^2}{(x-1)^3} = \ln x^2 - \ln(x-1)^3 = 2\ln x - 3\ln(x-1)$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad \log_a M^r = r\log_a M$$

Example

Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a \frac{\sqrt{x^2 + 1}}{x^3(x + 1)^4} \quad x > 0$$

as a sum and difference of logarithms. Express all powers as factors.

Solution

$$\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} = \log_a \sqrt{x^2+1} - \log_a[x^3(x+1)^4] \quad \text{Property (4)}$$

$$= \log_a \sqrt{x^2+1} - [\log_a x^3 + \log_a (x+1)^4] \quad \text{Property (3)}$$

$$= \log_a (x^2+1)^{1/2} - \log_a x^3 - \log_a (x+1)^4$$

$$= \frac{1}{2}\log_a (x^2+1) - 3\log_a x - 4\log_a (x+1) \quad \text{Property (5)}$$

Write a Logarithmic Expression as a Single Logarithm

Example

Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 7 + 4 \log_a 3$ (b) $\frac{2}{3} \ln 8 - \ln(5^2 - 1)$

(c) $\log_a x + \log_a 9 + \log_a(x^2 + 1) - \log_a 5$

Solution

(a) $$\log_a 7 + 4\log_a 3 = \log_a 7 + \log_a 3^4 \qquad r\log_a M = \log_a M^r$$
$$= \log_a 7 + \log_a 81$$
$$= \log_a(7 \cdot 81) \qquad \log_a M + \log_a N = \log_a(M \cdot N)$$
$$= \log_a 567$$

(b) $$\frac{2}{3}\ln 8 - \ln(5^2 - 1) = \ln 8^{2/3} - \ln(25 - 1) \qquad r\log_a M = \log_a M^r$$
$$= \ln 4 - \ln 24 \qquad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$$
$$= \ln\left(\frac{4}{24}\right) \qquad \log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$$
$$= \ln\left(\frac{1}{6}\right)$$
$$= \ln 1 - \ln 6$$
$$= -\ln 6 \qquad \ln 1 = 0$$

(c) $$\log_a x + \log_a 9 + \log_a(x^2 + 1) - \log_a 5 = \log_a(9x) + \log_a(x^2 + 1) - \log_a 5$$
$$= \log_a[9x(x^2 + 1)] - \log_a 5$$
$$= \log_a\left[\frac{9x(x^2 + 1)}{5}\right]$$

Theorem

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

If $M = N$, then $\log_a M = \log_a N$. **(7)**

If $\log_a M = \log_a N$, then $M = N$. **(8)**

Evaluate Logarithms Whose Base Is Neither 10 Nor *e*

Theorem

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \textbf{(9)}$$

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a} \qquad \textbf{(10)}$$

Example

Using the Change-of-Base Formula

Approximate:

(a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$

Round answers to four decimal places.

Solution

(a) $$\log_5 89 = \frac{\log 89}{\log 5} \approx \frac{1.949390007}{0.6989700043} \approx 2.7889$$

or

$$\log_5 89 = \frac{\ln 89}{\ln 5} \approx \frac{4.48863637}{1.609437912} \approx 2.7889$$

(b) $$\log_{\sqrt{2}} \sqrt{5} = \frac{\log \sqrt{5}}{\log \sqrt{2}} = \frac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \frac{\log 5}{\log 2} \approx 2.3219$$

or

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = \frac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \frac{\ln 5}{\ln 2} \approx 2.3219$$