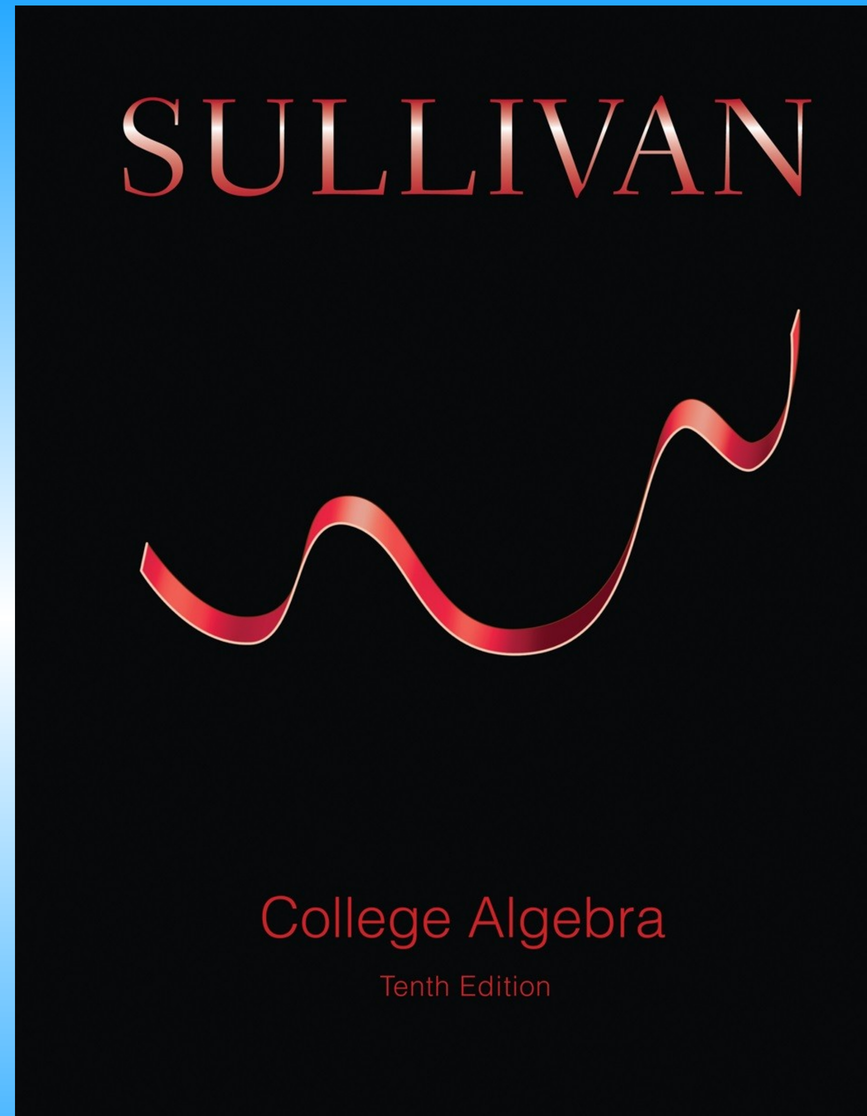
Chapter 6

Section 1



6.1 Composite Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

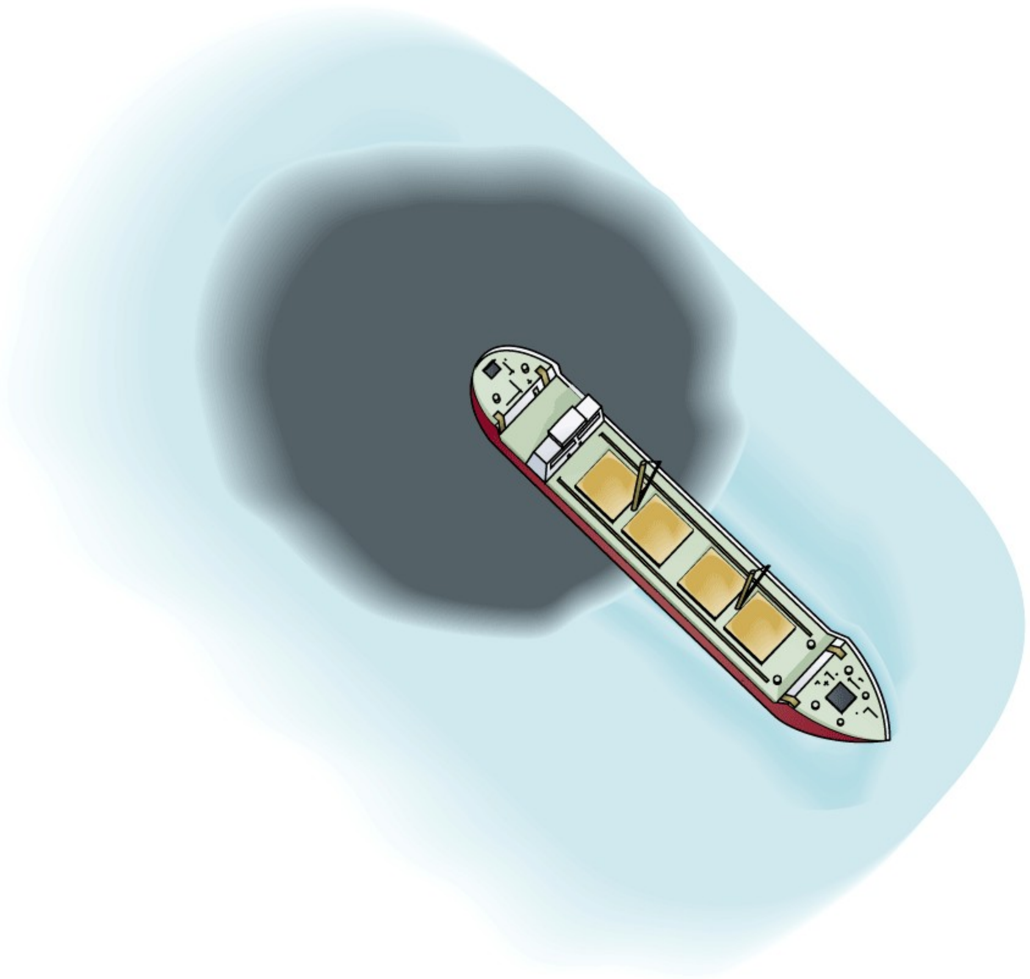
- Find the Value of a Function (Section 3.1, pp. 202–206)
- Domain of a Function (Section 3.1, pp. 206–208)

Now Work the **'Are You Prepared?'** problems on page 408.

OBJECTIVES
1. Form a Composite Function (p. 403)
2. Find the Domain of a Composite Function (p. 404)

Form a Composite Function

Figure



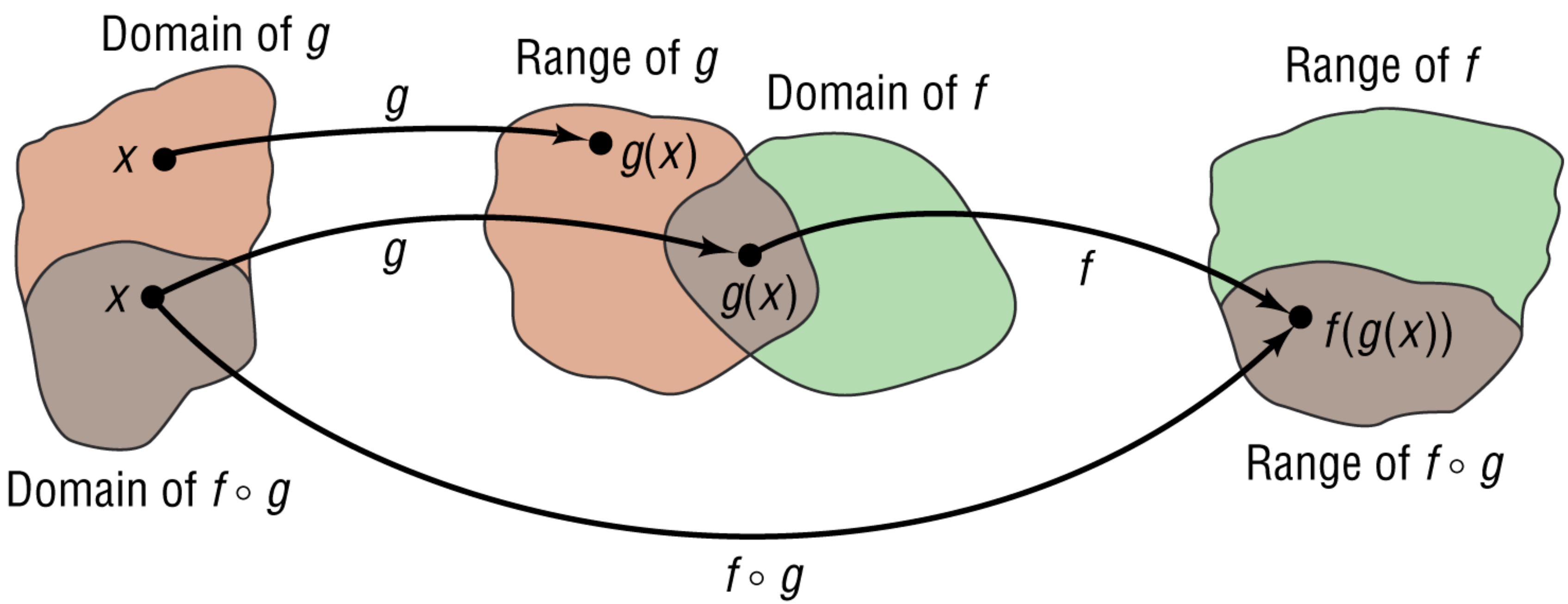
Definition

Given two functions f and g, the **composite function**, denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$(f \circ g)(x) = f(g(x))$$

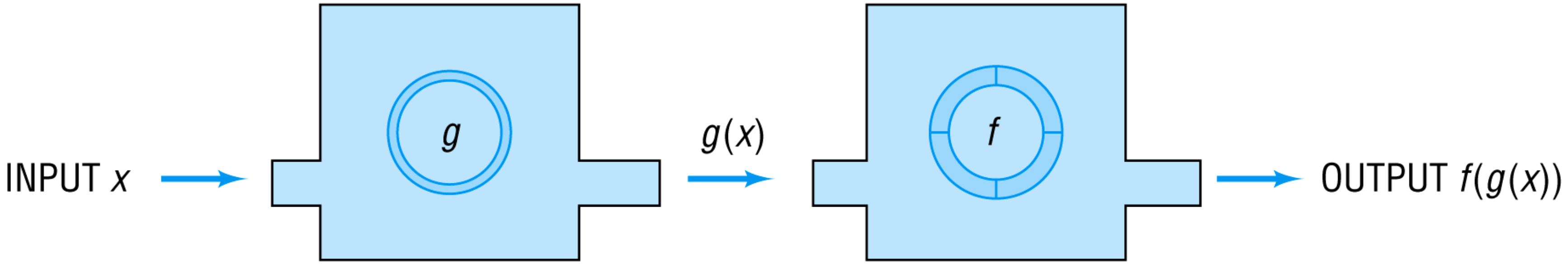
The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Figure



Domain of g
Range of g
Domain of f
Range of f
g
x
g(x)
g
x
g(x)
f
f(g(x))
Domain of f ∘ g
Range of f ∘ g
f ∘ g

Figure



Example

Evaluating a Composite Function

Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$. Find:

(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

Solution

(a) $(f \circ g)(1) = f(g(1)) = f(4) = 2 \cdot 4^2 - 3 = 29$

$g(x) = 4x$, $g(1) = 4$ $\qquad$ $f(x) = 2x^2 - 3$

(b) $(g \circ f)(1) = g(f(1)) = g(-1) = 4 \cdot (-1) = -4$

$f(x) = 2x^2 - 3$, $f(1) = -1$ $\qquad$ $g(x) = 4x$

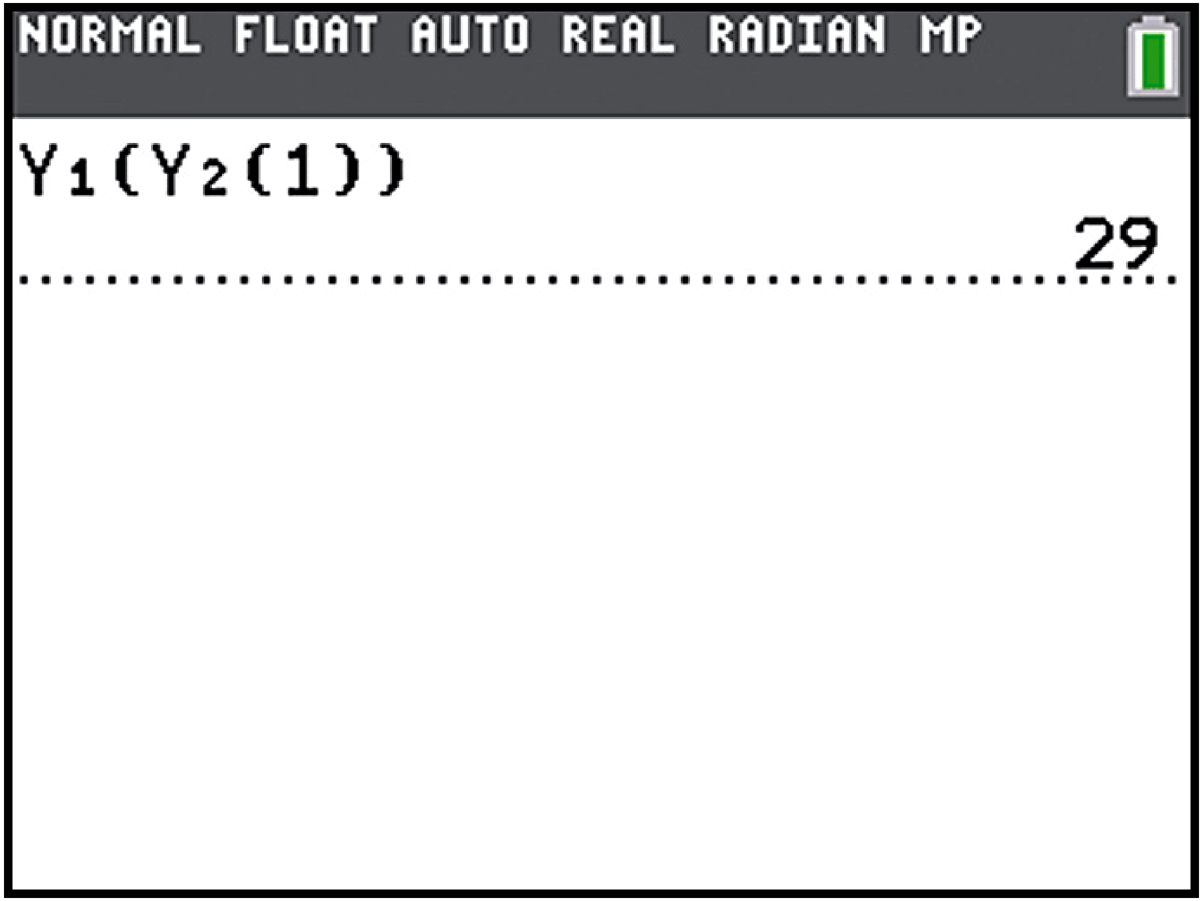
(c) $(f \circ f)(-2) = f(f(-2)) = f(5) = 2 \cdot 5^2 - 3 = 47$

$f(-2) = 2(-2)^2 - 3 = 5$

(d) $(g \circ g)(-1) = g(g(-1)) = g(-4) = 4 \cdot (-4) = -16$

$g(-1) = -4$

Figure



Find the Domain of a Composite Function

Example

Finding a Composite Function and Its Domain

Suppose that $f(x) = \dfrac{1}{x + 2}$ and $g(x) = \dfrac{4}{x - 1}$.

Find: (a) $f \circ g$ (b) $f \circ f$

Then find the domain of each composite function.

Solution

The domain of f is $\{x|x \neq -2\}$ and the domain of g is $\{x|x \neq 1\}$.

$$\text{(a)}\quad (f \circ g)(x) = f(g(x)) = f\left(\frac{4}{x-1}\right) = \frac{1}{\frac{4}{x-1}+2} = \frac{x-1}{4+2(x-1)} = \frac{x-1}{2x+2} = \frac{x-1}{2(x+1)}$$

$f(x) = \dfrac{1}{x+2}$ (first step); Multiply by $\dfrac{x-1}{x-1}$. (second step)

In Example 3, the domain of $f \circ g$ was found to be $\{x|x \neq -1, x \neq 1\}$.

Solution continued

The domain of $f \circ g$ also can be found by first looking at the domain of g: $\{x | x \neq 1\}$. Exclude 1 from the domain of $f \circ g$ as a result. Then look at $f \circ g$ and note that x cannot equal -1, because $x = -1$ results in division by 0. So exclude -1 from the domain of $f \circ g$. Therefore, the domain of $f \circ g$ is $\{x | x \neq -1, x \neq 1\}$.

(b) $$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2} = \frac{x+2}{1 + 2(x+2)} = \frac{x+2}{2x+5}$$

$\uparrow$ $f(x) = \dfrac{1}{x+2}$ $\qquad$ $\uparrow$ Multiply by $\dfrac{x+2}{x+2}$.

The domain of $f \circ f$ consists of all values of x in the domain of f, $\{x | x \neq -2\}$, for which

$$f(x) = \frac{1}{x+2} \neq -2 \qquad \frac{1}{x+2} = -2$$

$$1 = -2(x+2)$$

$$1 = -2x - 4$$

$$2x = -5$$

$$x = -\frac{5}{2}$$

Solution continued

or, equivalently,

$$x \neq -\frac{5}{2}$$

The domain of $f \circ f$ is $\left\{ x \middle| x \neq -\frac{5}{2}, x \neq -2 \right\}$.

The domain of $f \circ f$ also can be found by recognizing that -2 is not in the domain of f and so should be excluded from the domain of $f \circ f$. Then, looking at $f \circ f$, note that x cannot equal $-\frac{5}{2}$. Do you see why? Therefore, the domain of $f \circ f$ is $\left\{ x \middle| x \neq -\frac{5}{2}, x \neq -2 \right\}$.

Example

Showing That Two Composite Functions Are Equal

If $f(x) = 3x - 4$ and $g(x) = \frac{1}{3}(x + 4)$, show that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

for every x in the domain of $f \circ g$ and $g \circ f$.

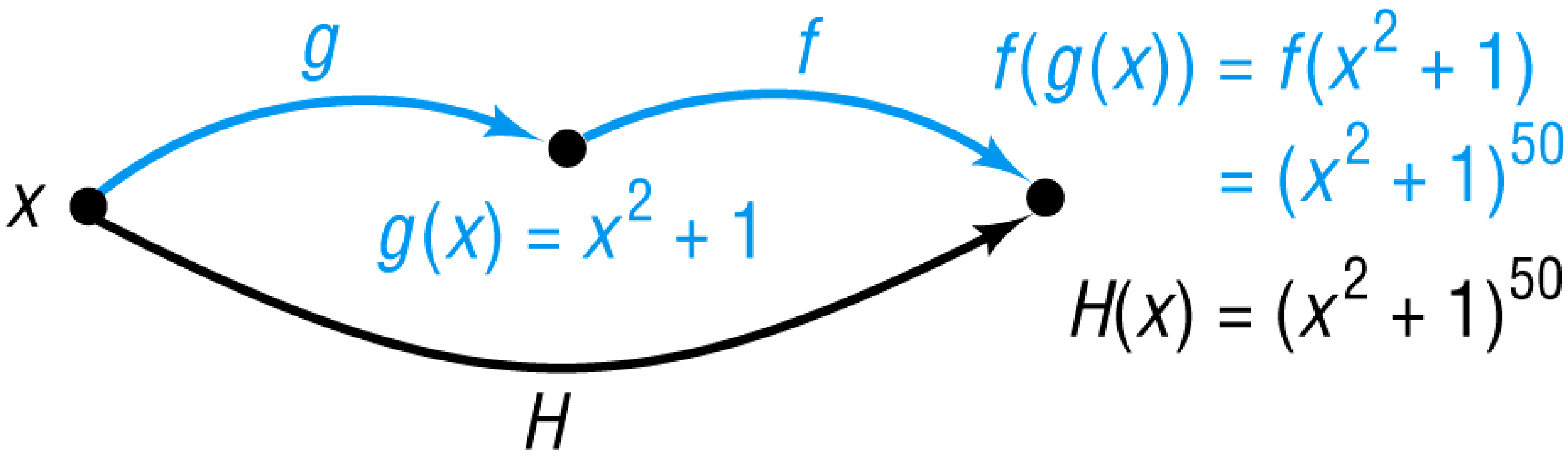
Solution

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{x+4}{3}\right) \qquad g(x) = \frac{1}{3}(x+4) = \frac{x+4}{3} \\
&= 3\left(\frac{x+4}{3}\right) - 4 \qquad f(x) = 3x - 4. \\
&= x + 4 - 4 = x
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(3x - 4) \qquad f(x) = 3x - 4 \\
&= \frac{1}{3}[(3x - 4) + 4] \qquad g(x) = \frac{1}{3}(x+4). \\
&= \frac{1}{3}(3x) = x
\end{aligned}
$$

We conclude that $(f \circ g)(x) = (g \circ f)(x) = x$.

Figure



Example

Finding the Components of a Composite Function

Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x+1}$.

Solution

Here H is the reciprocal of $g(x) = x + 1$. Let $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = \frac{1}{x + 1} = H(x)$$