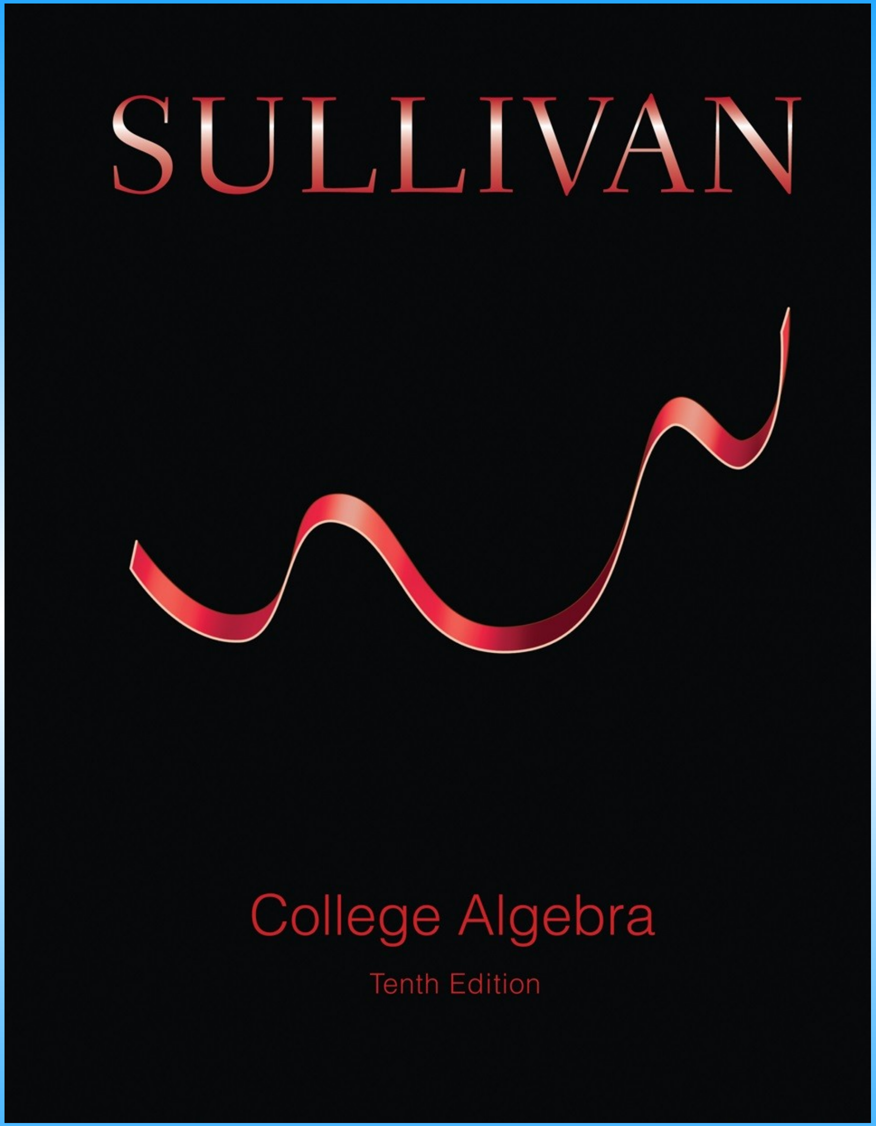
Chapter 2

Section 1



2.1 The Distance and Midpoint Formulas

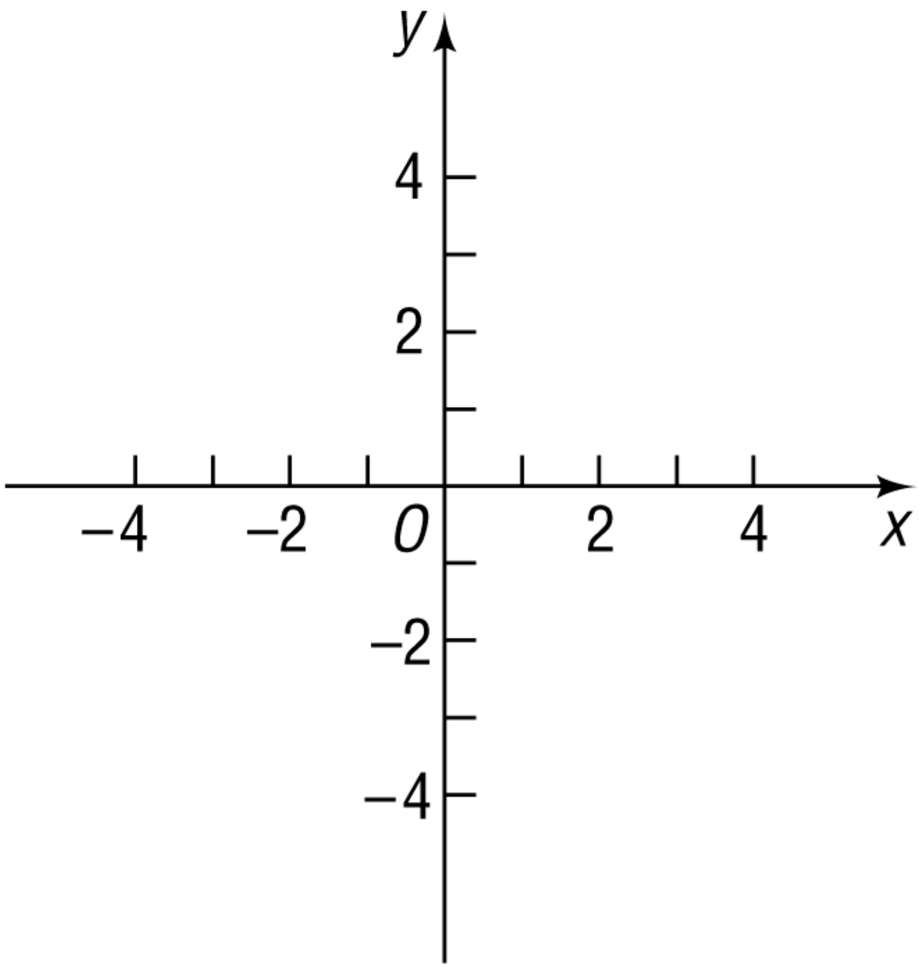
PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Algebra Essentials (Chapter R, Section R.2, pp. 17–26)
- Geometry Essentials (Chapter R, Section R.3, pp. 30–35)

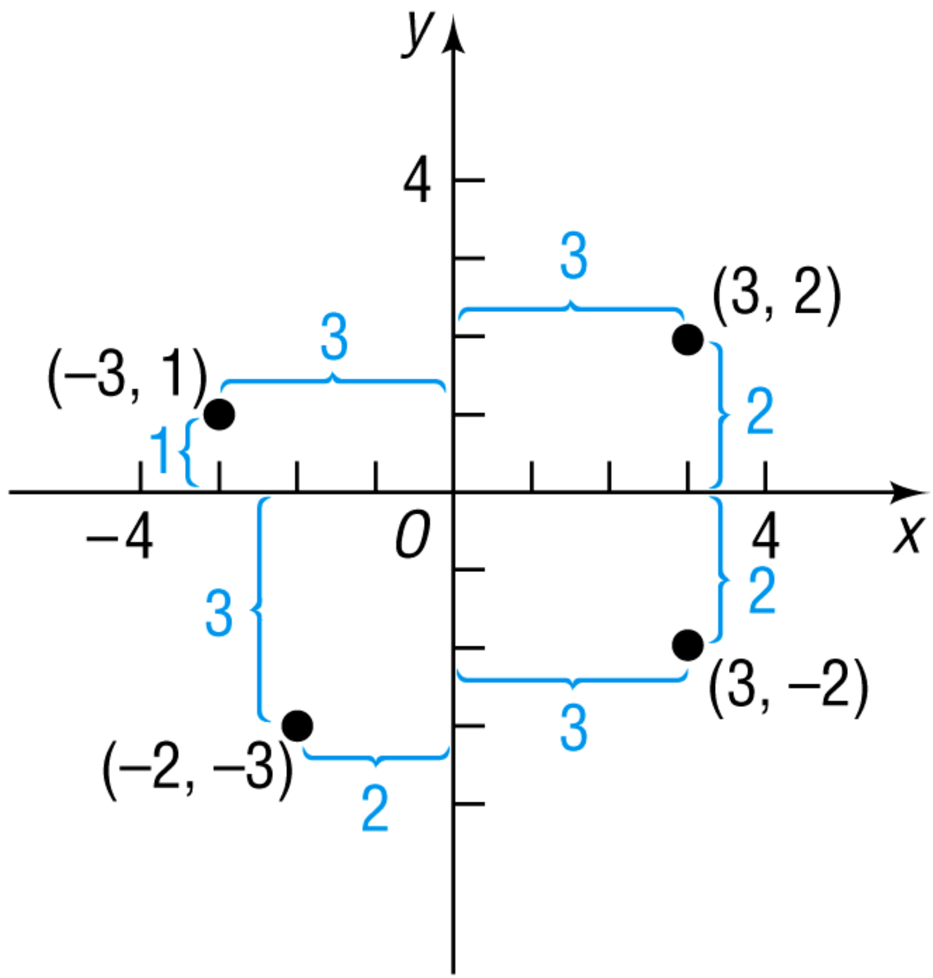
Now Work the **'Are You Prepared?'** problems on page 154.

OBJECTIVES
1. Use the Distance Formula (p. 151)
2. Use the Midpoint Formula (p. 153)

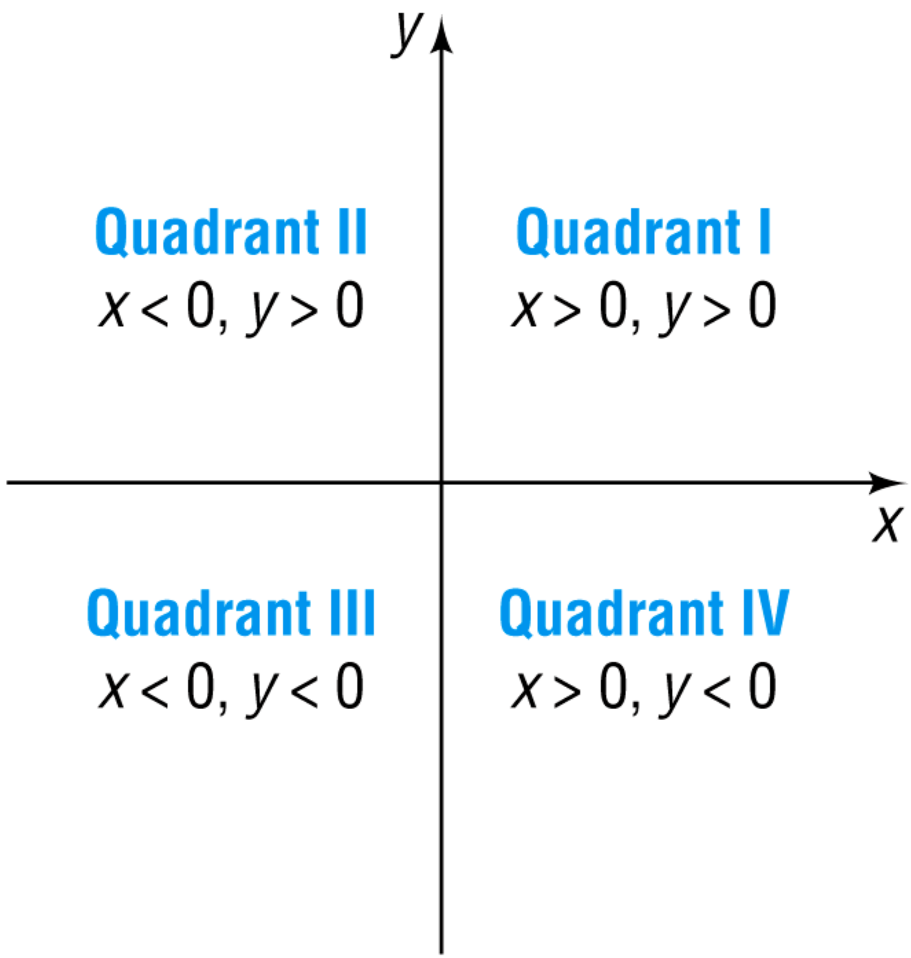
xy-Plane



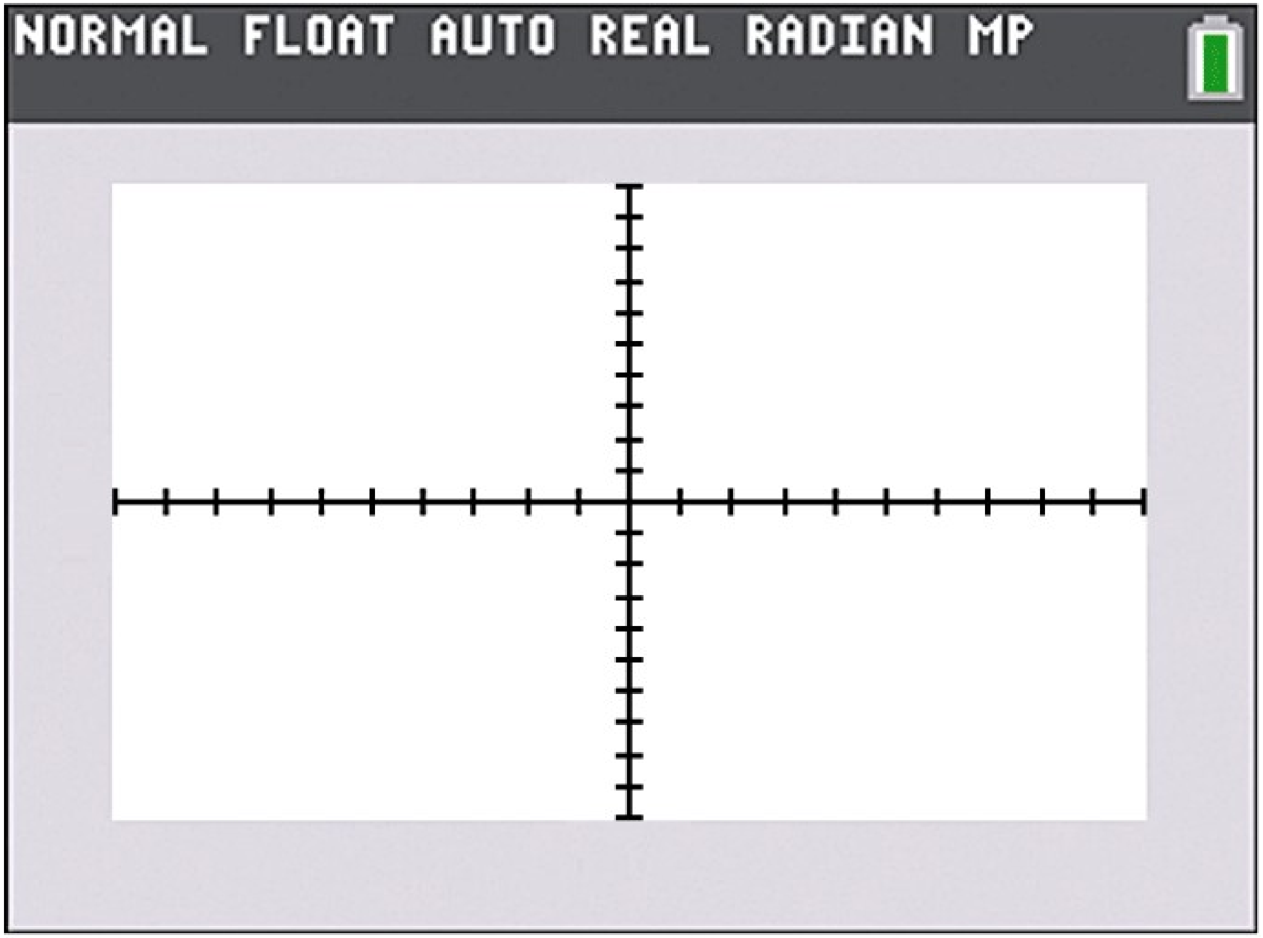
Some Plotted Points



Quadrants

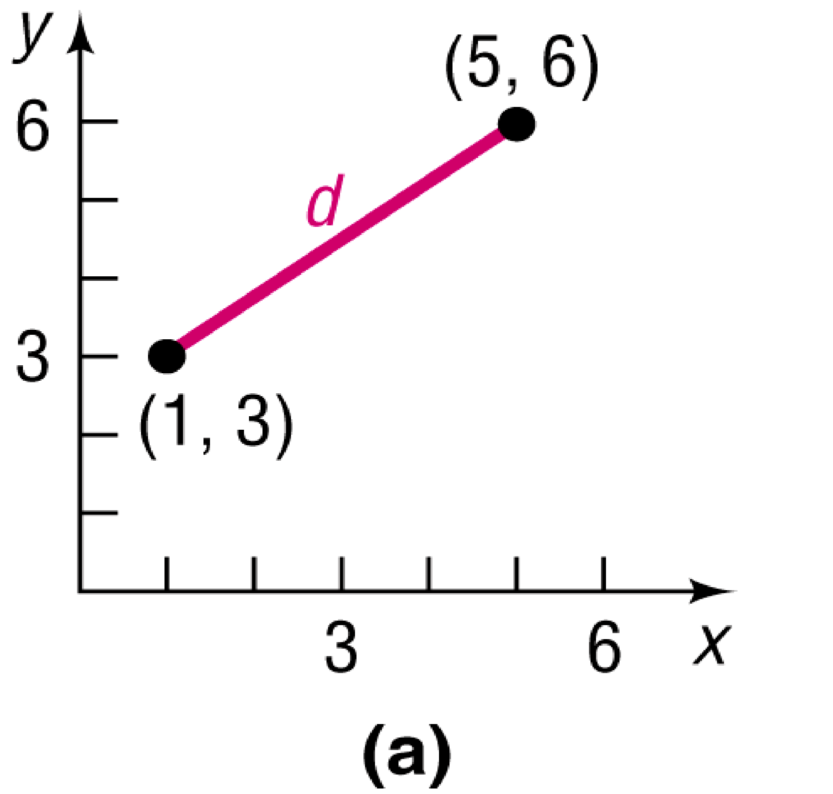

y
Quadrant II
x < 0, y > 0
Quadrant I
x > 0, y > 0
x
Quadrant III
x < 0, y < 0
Quadrant IV
x > 0, y < 0

TI-84 Plus C Standard Viewing Rectangle



Use the Distance Formula

Distance Between Two Points



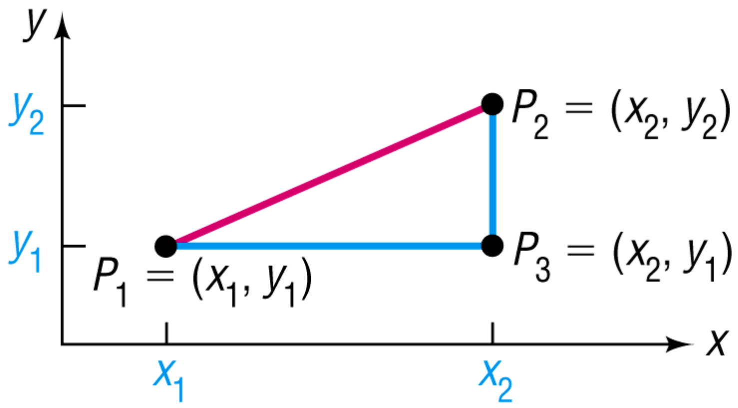
Theorem

Distance Formula

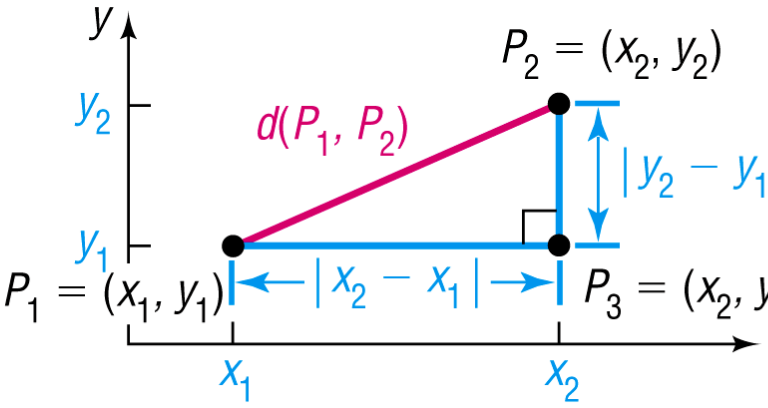
The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted by $d(P_1, P_2)$, is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \textbf{(1)}$$

Proof of Distance Formula

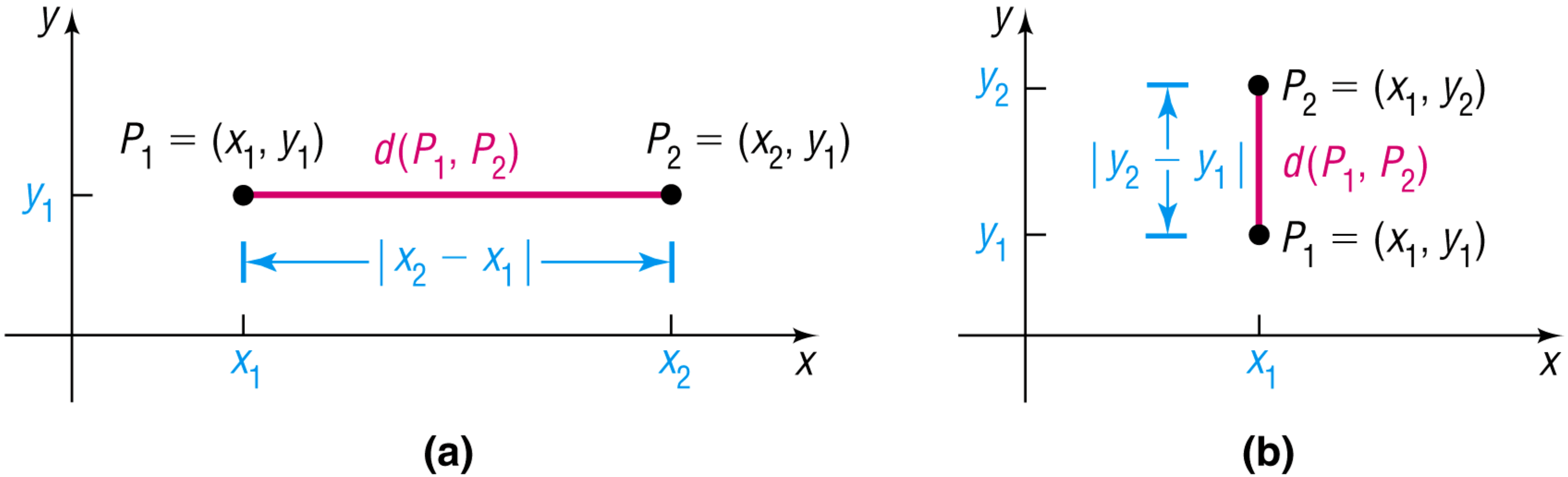


(a)



(b)

Proof continued



Example

Using the Distance Formula

Find the distance d between the points $(-4, 5)$ and $(3, 2)$.

Solution

Using the distance formula, equation (1), reveals that the distance d is

$$d = \sqrt{[3-(-4)]^2 + (2-5)^2} = \sqrt{7^2 + (-3)^2}$$
$$= \sqrt{49+9} = \sqrt{58} \approx 7.62$$

Example

Using Algebra to Solve Geometry Problems

Consider the three points $A = (-2, 1)$, $B = (2, 3)$, and $C = (3, 1)$.

(a) Plot each point and form the triangle ABC.

(b) Find the length of each side of the triangle.

(c) Show that the triangle is a right triangle.

(d) Find the area of the triangle.

Solution

(a) Figure 8 shows the points A, B, C and the triangle ABC.

(b) To find the length of each side of the triangle, use the distance formula, equation (1).

$$d(A,B) = \sqrt{[2-(-2)]^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$d(B,C) = \sqrt{(3-2)^2 + (1-3)^2} = \sqrt{1+4} = \sqrt{5}$$

$$d(A,C) = \sqrt{[3-(-2)]^2 + (1-1)^2} = \sqrt{25+0} = 5$$

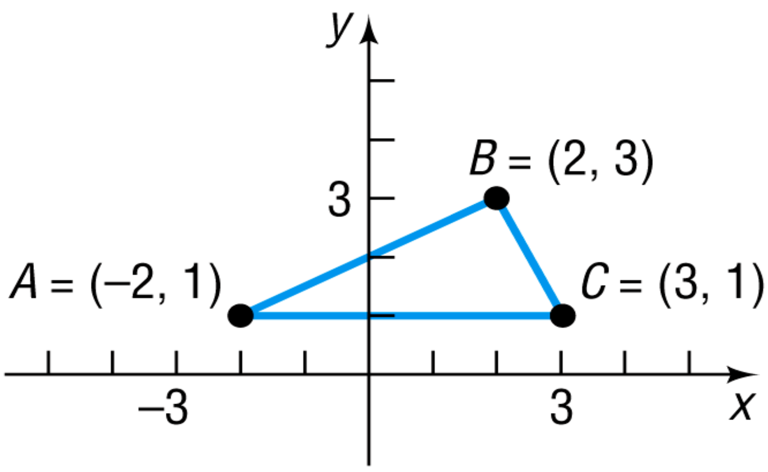


Figure 8

Solution continued

(c) If the sum of the squares of the lengths of two of the sides equals the square of the length of the third side, then the triangle is a right triangle. Looking at Figure 8, it seems reasonable to conjecture that the angle at vertex B might be a right angle. We shall check to see whether

$$[d(A,B)]^2 + [d(B,C)]^2 = [d(A,C)]^2$$

Using the results in part (b) yields

$$\begin{aligned}[d(A,B)]^2 + [d(B,C)]^2 &= \left(2\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 \\ &= 20 + 5 = 25 = [d(A,C)]^2\end{aligned}$$

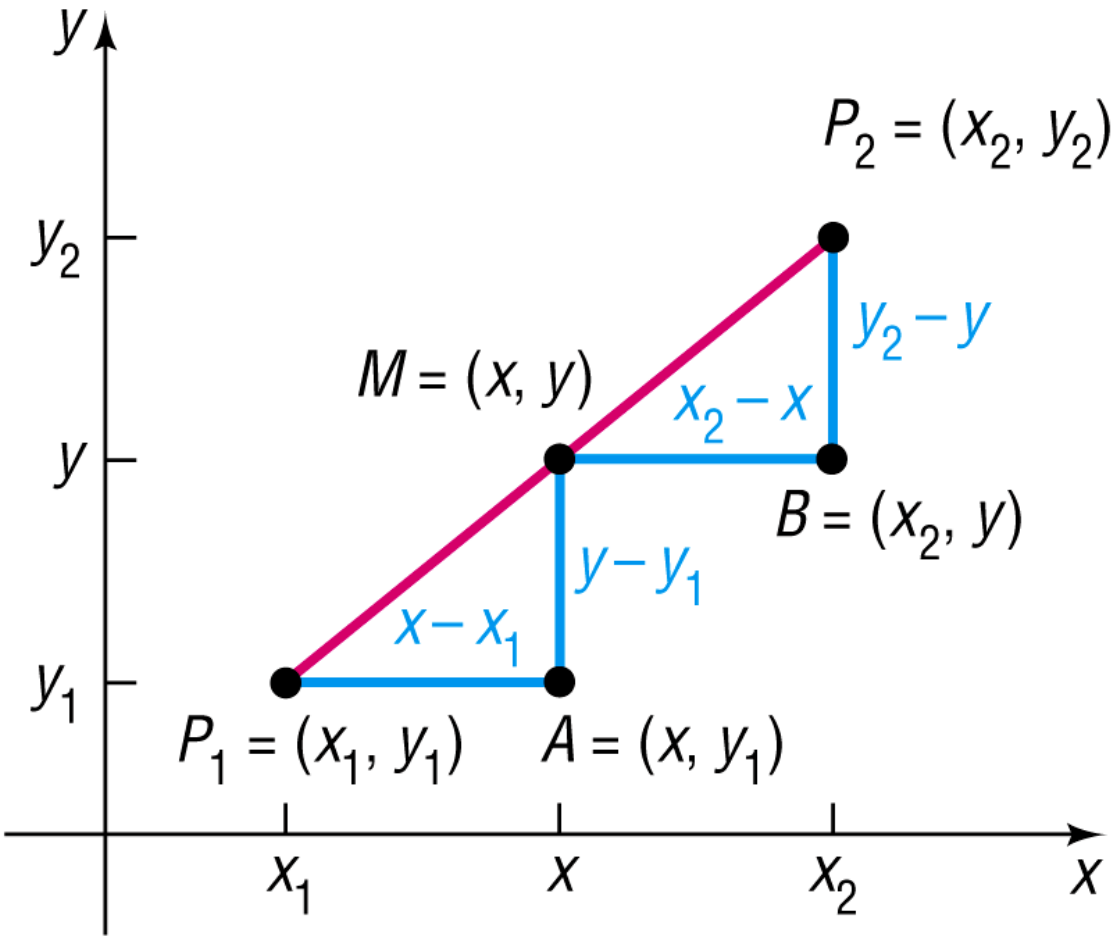
It follows from the converse of the Pythagorean Theorem that triangle ABC is a right triangle.

(d) Because the right angle is at vertex B, the sides AB and BC form the base and height of the triangle. Its area is

$$\text{Area} = \frac{1}{2}(\text{Base})(\text{Height}) = \frac{1}{2}\left(2\sqrt{5}\right)\left(\sqrt{5}\right) = 5 \text{ square units}$$

Use the Midpoint Formula

Derivation of Midpoint Formula



Theorem

Midpoint Formula

The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \textbf{(2)}$$

Example

Finding the Midpoint of a Line Segment

Find the midpoint of the line segment from $P_1 = (-5, 5)$ to $P_2 = (3, 1)$. Plot the points P_1 and P_2 and their midpoint.

Solution

Apply the midpoint formula (2) using $x_1 = -5, y_1 = 5, x_2 = 3$, and $y_2 = 1$. Then the coordinates (x, y) of the midpoint M are

$$x = \frac{x_1 + x_2}{2} = \frac{-5 + 3}{2} = -1 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{5 + 1}{2} = 3$$

That is, $M = (-1, 3)$. See Figure 10.

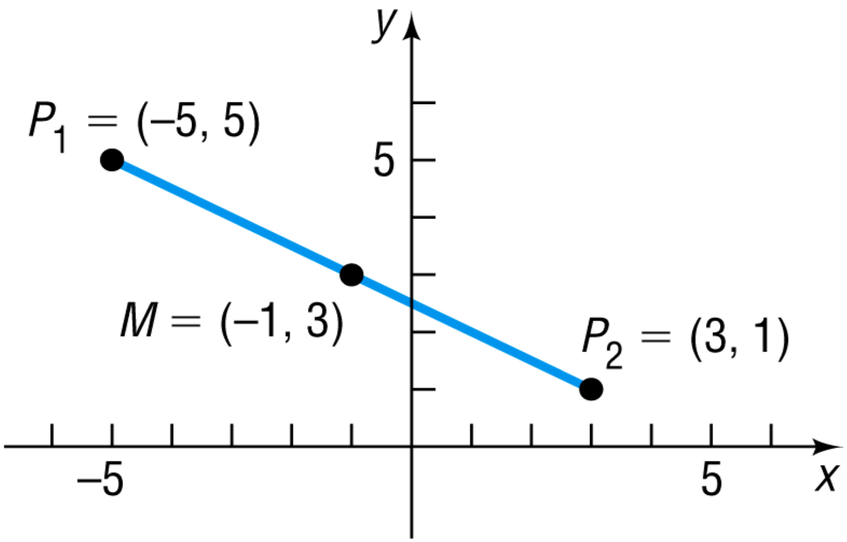


Figure 10