BUS 333: Finance I
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NOTES ON THE MATHEMATICS OF FINANCE

6. Present Value of an Annuity

In section 5 we defined annuities, and introduced the special notation for the present value of a standardized annuity:

$$a_{\overline{n}|i} = \frac{1 - v^n}{i}$$

In this section of the Notes on the Mathematics of Finance, I will go over the algebra which is used to derive the above formula. This is good material to master if you plan to make a career in finance or in the financial services industry.

The key to the present value formula is a result from high school algebra. Let's start by refreshing our memories of that result. Note the following:

$$\begin{aligned}
(1-x)(1+x) &= 1 + x - (x + x^2) = 1 - x^2 \\
(1-x)(1+x+x^2) &= 1 + x + x^2 - (x + x^2 + x^3) = 1 - x^3 \\
(1-x)(1+x+x^2+x^3) &= 1 + x + x^2 + x^3 - (x + x^2 + x^3 + x^4) = 1 - x^4
\end{aligned}$$

and so on. In general:

$$(1-x)(1 + x + x^2 + ... + x^{n-1}) = 1 - x^n$$

Don't just acept this result. Be sure you see how it was derived.

Now we devide both sides of the equation by $(1 - x)$ to get:

$$1 + x + x^2 + ... + x^{n-1} = \frac{1 - x^n}{1 - x}$$

This is the result we need from high school algebra. Now let's see how it is used to derive the formula for the present value of a standardized annuity with $n$ payments of \$1 at the end of each period, with a periodic interest rate of $i$.

The present value of the first payment is $1v = v$.

The present value of the second payment is $1v^2 = v^2$.

The present value of the third payment is $1v^3 = v^3$.

and so on. The present value of the last payment is $1v^n = v^n$.

Adding up the present value of each of the individual payments gives us the present value of the annuity.

$$a_{\overline{n}|i} = v + v^2 + v^3 + ... + v^n$$

Next, factor out a $v$ from each term on the right hand side (RHS).

$$a_{\overline{n}|i} = v(1 + v^2 + v^3 + ... + v^{n-1})$$

The expression in parentheses is in the same form as the result from high school algebra that we derived above. So we use that result here.

$$
\begin{aligned}
a_{\overline{n}|i} &= v(1 + v^2 + v^3 + ... + v^{n-1}) \\
a_{\overline{n}|i} &= v(\frac{1 - v^n}{1 - v}) = \frac{v}{1 - v}(1 - v^n)
\end{aligned}
$$

Here we use the fact that $v = \frac{1}{1+i}$, and reduce the expression $\frac{v}{1-v}$.

$$
\begin{aligned}
\frac{v}{1 - v} &= \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}} \\
&= \frac{\frac{1}{1+i}}{\frac{1+i}{1+i} - \frac{1}{1+i}} \\
&= \frac{\frac{1}{1+i}}{\frac{i}{1+i}} = \frac{1}{i}
\end{aligned}
$$

With this reduction, we replace $\frac{v}{1-v}$ with $\frac{1}{i}$ in the expression for the the present value of an annuity.

$$
\begin{aligned}
a_{\overline{n}|i} &= \frac{v}{1 - v}(1 - v^n) \\
&= \frac{1}{i}(1 - v^n) \\
&= \frac{1 - v^n}{i}
\end{aligned}
$$

This is the formula for the present value of a standard annuity. If you are able to follow the algebraic reasoning to develop this formula, then you are in good standing to master all the mathematics we will use in Finance 1. If not, then don't hesitate to come by in office hours to go over the particular steps that are giving you trouble.

Now, try this on your own. Derive a formula for the present value of an annuity of $n$ payments of \$1 starting right away, when the periodic interest rate is $i$ per period. In notation, find the formula for:

$$\ddot{a}_{\overline{n}|i} = ???$$