BUS 333: Finance I
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NOTES ON THE MATHEMATICS OF FINANCE

5. **Annuity Notation**

Suppose you just won a $1 million dollar lottery prize. You have a choice to receive payments of $50,000 at the end of each year for 20 years, or the present value of those payments. If you can invest at an interest rate of 5%, what is the least lump-sum amount that you would accept in lieu of the 20 annual payments?

Well, it's a little bit of work to figure this out, but with the tools we have learned so far, we can do this with the help of an EXCEL spreadsheet. We will calculate the present value of each one of the payments, and add up to get the total present value. See the worksheet below.

Year	Payment	v^t	Present Value = Payment x v^t		
1	50,000	0.952381	47,619.05		
2	50,000	0.907029	45,351.47		
3	50,000	0.863838	43,191.88		
4	50,000	0.822702	41,135.12		
5	50,000	0.783526	39,176.31		
6	50,000	0.746215	37,310.77		
7	50,000	0.710681	35,534.07		
8	50,000	0.676839	33,841.97		
9	50,000	0.644609	32,230.45		
10	50,000	0.613913	30,695.66		
11	50,000	0.584679	29,233.96		
12	50,000	0.556837	27,841.87		
13	50,000	0.530321	26,516.07		
14	50,000	0.505068	25,253.40		
15	50,000	0.481017	24,050.85		
16	50,000	0.458112	22,905.58		
17	50,000	0.436297	21,814.83		
18	50,000	0.415521	20,776.03		
19	50,000	0.395734	19,786.70		
20	50,000	0.376889	18,844.47		
Total Present Value			623,110.52		

So the present value of the million dollar prize is $623,110.52, before taxes. (With a 40% tax rate, a reasonable estimate, the prize is worth only $373,866.31. Not bad, but don't quit your day job.)

The point of the example is not about the shrinking value of the million dollar prize. The point is about how to value periodic payments. With our basic tools, we can value the stream of cash flows one at a time. With the help of an EXCEL spreadsheet, it is doable. In this section of Notes on the Mathematics of Finance, we will learn about special

notation that has been developed for these frequently encountered situations where payments occur on a regular periodic basis. This will facilitate writing mathematical expressions in financial problems that involve periodic payments. Then we will learn how to evaluate the notation mathematically.

Annuity Notation

An annuity is a series of regular periodic payments, such as the annual payment of the lottery prize in the example above. Other examples of an annuity are payments made on a mortgage, or car payments, or even a pension. (A pension is a special type of annuity, called a life annuity, since it continues for life, not just for a certain period of time.) Note that an annuity does not have to be annual payments. In fact, annuities are often not annual, but monthly or some other period. But they always involve payments of a regular amount on a periodic schedule.

With this idea in mind, let's introduce some special notation to represent the <u>present value of a standardized annuity of \$1 per period for "n" periods, calculated at an interest rate of i</u>. The cash flows of this standard annuity are shown below.

Cash Flow   1   1   1   1   1
    |_____|_____|_____|____/ /_____|_____|____

Time 0   1   2   3   *n-1*   *n*
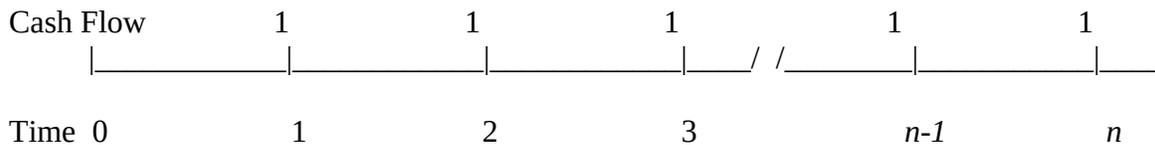
Note that there are "*n*" cash flows of \$1 each; each payment occurs at the <u>end</u> of the period. We will denote the present value of "*n*" standardized cash flows by the notation $a_{\overline{n}|\_i}$. In the notation is everything we need to calculate the present value. There is the interest rate *i*, the number of payments *n,* and the knowledge that the payment is a standard \$1.

Suppose the payments were \$50, instead of \$1, and they were payable for 20 years, at an interest rate of 6%. Then the present value of the payments is simply $\$50 \times a_{\overline{20}|.06}$. (We usually skip the times notation and just write $\$50 a_{\overline{20}|.06}$.) Our notation for the present value of the cash flows contains within it everything we need to calculate the present value. We see the amount of each payment, the number of payments, and the interest rate at which to discount the future payments.

There are many examples of annuities in everyday financial transactions. For example, car lease payments are an annuity, as are mortgage payments, rent, and student loan payments. It will be very helpful to have an easy way to represent the present value of an annuity, with all the information to calculate the value easily at hand.

Now it's time to practice, and get familiar with the notation.   What is the cash flow whose present value is represented in the following expressions?

1. $2500a_{\overline{240}|.005}$

(This is the present value of 240 payments of 2500, when interest rates are .5% per period. For example, it could represent the present value of a 20 year mortgage with an interest rate of 6%, compounded monthly.)

2. $40a_{\overline{20}|.05}$

(This is the present value of 20 payments of $40, when interest rates are 5% per period. For example, it could represent the present value of $40 interest coupons on a 10 year corporate bond, evaluated when the interest rate environment is 10%, compounded semi-annually.)

3. $250a_{\overline{120}|.0025}$

(This is the present value of 120 payments of $250, calculated with an interest rate of ¼% per period. For example, it might be the present value of your student loans, paid off over 10 years at an interest rate of 3%, compounded monthly.)

You get the idea. The annuity notation allows us to express a complex stream of payments in a mathematical expression, which contains all the information we need to evaluate it.

Calculating the Present Value of an Annuity

For the standard annuity, the formula to calculate the present value that it represents is:

$$a_{\overline{n}|\_i} = \frac{1 - v^n}{i}$$

Let's calculate a few values, just to get a feel for the way the present value of an annuity behaves.

# Payments	Periodic Interest Rate	v^n	(1-v^n)/i			
1	0.05	0.952381	0.952381			
2	0.05	0.907029	1.85941			
3	0.05	0.863838	2.723248			
4	0.05	0.822702	3.545951			
5	0.05	0.783526	4.329477			
10	0.05	0.613913	7.721735			
20	0.05	0.376889	12.46221			
30	0.05	0.231377	15.37245			
1	0.12	0.892857	0.892857			
2	0.12	0.797194	1.690051			
3	0.12	0.71178	2.401831			
4	0.12	0.635518	3.037349			
5	0.12	0.567427	3.604776			
10	0.12	0.321973	5.650223			
20	0.12	0.103667	7.469444			
30	0.12	0.033378	8.055184			

The value in the last column is the value of the annuity expression $a_{\overline{n}|\_i} = \dfrac{1 - v^n}{i}$. To show you how it is calculated, I show in the third column the intermediate value of v^n, so you can reproduce the value easily. Note that an annuity with only 1 payment has a present value equal to v, which should be obvious. As the number of payments increases, the present value of the annuity increases by the present value of the additional payment. In every case, the present value of the annuity is less than the number of payments. The further out in time the payment is made, the less impact it will have on the present value. See that the annuity value for 20 payments is $12.46221, and the present value of 30 payments is $15.37245. The extra 10 payments only added $2.91024 to the present value. When interest rates are even higher, such as in the 12% example in the table above, those extra 10 payments from 20 to 30 adds only 58.574 cents to the present value. (It increased from $7.46944 to $8.055184)

Let's go back to the $1 million Lottery Prize in the opening example. Using our annuity notation, the present value of the prize is: $PV = \$50,000 a_{\overline{20}|.05}$. To evaluate this expression we first calculate $v^{20} = .952381^{20} = .37689$. Now, using the formula for the calculation of an annuity, we get:

$$50,000 a_{\overline{20}|.05} = 50,000\frac{\left(1 - v^{20}\right)}{i} = 50,000\frac{1 - .37689}{.05} = 50,000 \times 12.4622 = \$623,110.10.$$

Except for some rounding errors in the pennies, this is the value we got from the EXCEL spreadsheet in the opening paragraph.

Suppose interest rates change and instead of 5% interest, the discounted value of the prize payments is calculated at 5.8% interest. What is the prize worth in that case?

$$50,000 a_{\overline{20}|.058} = 50,000 \frac{\left(1 - v^{20}\right)}{i} = 50,000 \frac{1 - .323807}{.058} = 50,000 \times 11.658493 = \$582,924.65$$

Note that at higher interest rates, those future payments are worth less today. That should be intuitively reasonable to you. After all, if interest rates are high, it means that the people giving out the prize need to set aside less money today to have enough on hand in the future to make the payments when they are due.

Be sure to work through this example, and get the same answer. We will move on to more complicated examples quickly.

Annuities Due

Oftentimes, the first payment of the annuity is due right away, at the beginning of the period, not waiting to the end of the period as is the case in our standardized annuity. This is called an annuity due, which name you can forget about. Just call it an annuity whose payments begin right away. This might be the case with car payments, or with your mortgage, or whatever. Relax. We don't need to change much. We need to adjust our notation to distinguish annuities that have payments at the beginning of the period from the standard case annuity in which payments are at the end of the period. Annuities due are denoted as follows: $\ddot{a}_{\overline{n}|\_i}$. The umlaut over the *a* alerts us that this is a case where payments are made at the beginning of the period, not the end.

It is very easy to calculate the present value of an annuity due, using the standard annuity. Suppose you are trying to find the present value of an annuity which makes 10 payments beginning right away, at *i* interest. That is the same as the present value of a standard annuity that makes 9 payments at the end of the year, plus 1 for the first payment that is made right away. In mathematical symbols, $\ddot{a}_{\overline{10}|\_i} = 1 + a_{\overline{9}|\_i}$. Note that on both sides of this equation there are 10 payments of one.

I think this is enough for now. Next section we will tackle some problems, and practice working with the notation.