BUS 333: Finance I
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NOTES ON THE MATHEMATICS OF FINANCE

2. Interest Compounded More Frequently Than Annually

Compound interest does not have to compound on only an annual cycle. It is common for savings accounts, for example, to compound interest quarterly, monthly, or even daily. In a bank where interest is compounded quarterly, interest earned this quarter would itself earn interest in all subsequent quarters. How do we handle the situation where interest is compounded more frequently than annually?

The first thing is to nail down what meaning to give the expression "the interest rate is 8%, compounded quarterly". What exactly does that mean? The interest rate is stated as 8%, but the account does not earn 8% interest per year. What it means is that each calendar quarter, the bank will pay 2% (one quarter of 8%) of the account value at the end of the quarter as an interest payment. The yearly rate is 8%, but the actual rate is 2% per quarter. Over the course of a year, the bank will pay interest of 2% four times. But that does not result in 8% interest. Because the interest is compounded each quarter, the bank actually pays more than 8% interest on the account value at the beginning of the year. Let's look at the details.

Calendar Quarter	Quarterly Rate	Beginning Balance	Interest	Ending Balance		
1	2%	1000.00	20.00	1020.00		
2	2%	1020.00	20.40	1040.40		
3	2%	1040.40	20.81	1061.21		
4	2%	1061.21	21.22	1082.43		

Over the course of four calendar quarters, one year, the bank pays $82.43 in interest. In effect, the depositor earned 8.243% interest, not the stated 8% rate. But for clarity of meaning, the interest rate is stated as "8%, compounded quarterly". We call this stated 8% the "nominal" rate of interest, or sometimes it is called the annual percentage rate (APR). The actual resulting outcome, 8.243% is called the "annual effective interest rate" , since that is the effect of compounding quarterly. Note that the annual effective interest rate is simply the quarterly rate, compounded four times.

Mathematically, $1000 \times (1+.08243) = 1000 \times (1+.02)^4$. The left hand side is the value of a deposit after one year at the effective annual interest rate of 8.243%, and the right hand side is the value of a deposit after one year at 8%, compounded quarterly.

Here are two more common examples of interest rated compounded more frequently than annually.

Residential mortgages are usually paid monthly. It is convenient therefore to use an interest rate that is compounded monthly. What is the effective annual interest rate on a mortgage that has a nominal rate of 6%, compounded monthly? (Answer: 6.168%)

(Don't skip checking that the answer is correct!)

Corporate Bonds usually pay interest semiannually, at a stated nominal yearly rate, called the coupon rate. For example, suppose Ford Motor Company came to the market with an 8% bond issue. This means that investors could purchase bonds which will mature for $1000 at a specified date in the future, and that every six months up to the maturity date, Ford will make an interest coupon payment of $40. What is the effective annual interest rate on the bond? (Answer: 8.16%) [Note: For the purchaser of the bond to actually realize an 8.16% return on the initial investment, the purchaser must also be able to reinvest each coupon to earn 8%, compounded semiannually. This is unlikely. But for our purposes, we will ignore such real world complications, and calculate the effective interest rate as if the coupons were so reinvested. This reinvestment issue is not encountered in the case of a bank deposit, since the interest payments are in fact reinvested.]

By now you should have a good understanding of interest that is compounded more frequently than annually. So how do we compute future values in such a situation?

<u>Basic Example</u>
Suppose a bank pays interest at 8%, compounded quarterly. What would be the account balance on a $1000 deposit after 3 years?

We saw in the chart above that after one year, the account value is $1082.43, or $1000 \times (1+.02)^4$. In three years time, interest is compounded 12 quarters, so the future value after 12 quarters will be: $1000 \times (1+.02)^{12}$, which is $1268.24.

The solution requires that we look at the problem according to the frequency at which interest is compounded. If interest is compounded quarterly, then we need to know the interest rate <u>per quarter</u> (2% per quarter in the basic example), and the number of quarters to the point of the future value (in the basic example there were 12 quarters to the end of three years). Then we can calculate the future value just as we did for annual interest rates in Section 1. The mathematical expression is:

$$FV_t = (1+i)^n \times P$$

where *i* is the interest rate per period, *n* is the number of periods until the valuation point in time, *t*. Let's look at some practice examples.

Practice Examples

What is the value of a $1000 deposit …

a. … in 5 years at 4% compounded quarterly?
(Answer: $ 1,220.19) [$FV_5 = (1+.01)^{20} \times 1000$]

b. …in 10 years at 6% compounded monthly?
(Answer: $ 1,819.40) [$FV_{10} = (1+.005)^{120} \times 1000$]

c. …in 6 and ½ years, at 8% compounded semiannually?
(Answer: $ 1,665.07) [$FV_{6.5} = (1+.04)^{13} \times 1000$]

d. …in 3 years, at 6% compounded daily? [Note: To solve this problem, we need to know haw many days are in a year. It is common, but not universal, to assume 360 daily compounding periods in a year. Thus the interest rate per day is .06 divided by 360, or .000167.)
(Answer: $ 1,197.63) [$FV_3 = (1+.000167)^{1080} \times 1000$]

Interest is almost always compounded more frequently than annually, so you must be completely comfortable with how to solve this basic problem.

The mathematical expression for the future value at time *t* of a deposit *P*, which is earning interest that is compounded *m* times per year at a nominal rate of $i^{(m)}$, using the notation of finance is:

$$FV_t = (1+\frac{i^{(m)}}{m})^{t \times m} \times P$$

So, for a given time *t*,

(Semiannually) $$FV_t = (1+\frac{i^{(2)}}{2})^{t \times 2} \times P$$

(Quarterly) $$FV_t = (1+\frac{i^{(4)}}{4})^{t \times 4} \times P$$

(Monthly) $$FV_t = (1+\frac{i^{(12)}}{12})^{t \times 12} \times P$$

(Daily) $$FV_t = (1+\frac{i^{(360)}}{360})^{t \times 360} \times P$$

It looks worse than it is. Just think carefully about what you are doing, it will eventually make sense, and you will come out ok.

Finance I

Problem Set 1

1. You earn $2000 over the summer, above what you need to pay your school bills. You decide to put the money in the bank, and keep it there until you graduate in three years. If the bank is paying 4.75% interest annually, how much will you have at graduation (exactly three years from now).

2. Now, suppose the bank is paying 4.75%, compounded quarterly. How much will you have at graduation?

3. Suppose graduation is not three years away, but is two years and 9 months away. How much will you have at graduation?

4. What if graduation is 2 years and 10 months away? How much will you have at graduation?

5. Now, let's go back to the simple case of problem 1. Graduation is in exactly three years, and the bank is paying interest at 4.75% annually. You have $2000 extra to deposit, but you would like to use some of the money now and still save enough so that you have a nest egg of $1000 at graduation to start your new career. How much should you save now, and how much can you spend?