Mathematics

Numbers and Operations

Numbers and their classifications

Numbers are the basic building blocks of mathematics. Specific features of numbers are identified by the following terms:
Integers – The set of whole positive and negative numbers, including zero. Integers do not include fractions $\left(\frac{1}{3}\right)$, decimals (0.56), or mixed numbers $\left(7\frac{3}{4}\right)$.
Prime number – A whole number greater than 1 that has only two factors, itself and 1; that is, a number that can be divided evenly only by 1 and itself.
Composite number – A whole number greater than 1 that has more than two different factors; in other words, any whole number that is not a prime number. For example: The composite number 8 has the factors of 1, 2, 4, and 8.
Even number – Any integer that can be divided by 2 without leaving a remainder. For example: 2, 4, 6, 8, and so on.
Odd number – Any integer that cannot be divided evenly by 2. For example: 3, 5, 7, 9, and so on.
Decimal number – a number that uses a decimal point to show the part of the number that is less than one. Example: 1.234.
Decimal point – a symbol used to separate the ones place from the tenths place in decimals or dollars from cents in currency.
Decimal place – the position of a number to the right of the decimal point. In the decimal 0.123, the 1 is in the first place to the right of the decimal point, indicating tenths; the 2 is in the second place, indicating hundredths; and the 3 is in the third place, indicating thousandths.

The decimal, or base 10, system is a number system that uses ten different digits $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$. An example of a number system that uses something other than ten digits is the binary, or base 2, number system, used by computers, which uses only the numbers 0 and 1. It is thought that the decimal system originated because people had only their 10 fingers for counting.

Rational, irrational, and real numbers can be described as follows:
Rational numbers include all integers, decimals, and fractions. Any terminating or repeating decimal number is a rational number.
Irrational numbers cannot be written as fractions or decimals because the number of decimal places is infinite and there is no recurring pattern of digits within the number. For example, pi (π) begins with 3.141592 and continues without terminating or repeating, so pi is an irrational number.
Real numbers are the set of all rational and irrational numbers.

Factors and multiples

Factors are numbers that are multiplied together to obtain a product. For example, in the equation $2 \times 3 = 6$, the numbers 2 and 3 are factors. A prime number has only two factors (1 and itself), but other numbers can have many factors.

A common factor is a number that divides exactly into two or more other numbers. For example, the factors of 12 are 1, 2, 3, 4, 6, and 12, while the factors of 15 are 1, 3, 5, and 15. The common factors of 12 and 15 are 1 and 3.

A prime factor is also a prime number. Therefore, the prime factors of 12 are 2 and 3. For 15, the prime factors are 3 and 5.

The greatest common factor (GCF) is the largest number that is a factor of two or more numbers. For example, the factors of 15 are 1, 3, 5, and 15; the factors of 35 are 1, 5, 7, and 35. Therefore, the greatest common factor of 15 and 35 is 5.

> **Review Video:** **Greatest Common Factor (GCF)**
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The least common multiple (LCM) is the smallest number that is a multiple of two or more numbers. For example, the multiples of 3 include 3, 6, 9, 12, 15, etc.; the multiples of 5 include 5, 10, 15, 20, etc. Therefore, the least common multiple of 3 and 5 is 15.

Rounding

Rounding is the approximation of a number by decreasing or increasing it to the nearest possible exact value of the cutoff digit. This is done by the following steps:

1. Check the digit immediately to the right of the cutoff digit.
    a. If this digit is 5 or higher, add 1 to the cutoff digit.
    b. If this digit is 4 or lower, keep the original cutoff digit.
2. Eliminate all digits to the right of the cutoff digit.

For example, suppose we want to round the number 123.4567 to the nearest hundredth. The cutoff digit then is the 5. Immediately to the right is a 6, so we'll add 1 to the cutoff digit, making it a 6. The rounded number then is 123.46.

A number should only ever be rounded once. If we were to round our number above to the nearest tenth, the result would be 123.5. If we then tried to round it a second time, to the nearest integer, we would get 124. This is not proper rounding because rounding the original number to the nearest integer gives us 123.

A number should also not be rounded in the middle of a series of calculations; only at the end. Rounding in the middle tends to compound what is known as rounding error.

Operations and Algebraic Thinking

Operations

There are four basic mathematical operations:
Addition increases the value of one quantity by the value of another quantity. Example: $2 + 4 = 6; 8 + 9 = 17$. The result is called the sum. With addition, the order does not matter. $4 + 2 = 2 + 4$.

Subtraction is the opposite operation to addition; it decreases the value of one quantity by the value of another quantity. Example: $6 - 4 = 2; 17 - 8 = 9$. The result is called the difference. Note that with subtraction, the order does matter. $6 - 4 \neq 4 - 6$.
Multiplication can be thought of as repeated addition. One number tells how many times to add the other number to itself. Example: 3×2 (three times two) $= 2 + 2 + 2 = 6$. With multiplication, the order does not matter. $2 \times 3 = 3 \times 2$ or $3 + 3 = 2 + 2 + 2$.
Division is the opposite operation to multiplication; one number tells us how many parts to divide the other number into. Example: $20 \div 4 = 5$; if 20 is split into 4 equal parts, each part is 5. With division, the order of the numbers does matter. $20 \div 4 \neq 4 \div 20$.

An exponent is a superscript number placed next to another number at the top right. It indicates how many times the base number is to be multiplied by itself. Exponents provide a shorthand way to write what would be a longer mathematical expression. Example: $a^2 = a \times a$; $2^4 = 2 \times 2 \times 2 \times 2$. A number with an exponent of 2 is said to be "squared," while a number with an exponent of 3 is said to be "cubed." The value of a number raised to an exponent is called its power. So, 8^4 is read as "8 to the 4th power," or "8 raised to the power of 4." A negative exponent is the same as the reciprocal of a positive exponent. Example: $a^{-2} = \frac{1}{a^2}$.

> ➤ **Review Video: <u>Exponents</u>**
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Order of Operations is a set of rules that dictates the order in which we must perform each operation in an expression so that we will evaluate it accurately. If we have an expression that includes multiple different operations, Order of Operations tells us which operations to do first. The most common mnemonic for Order of Operations is PEMDAS, or "Please Excuse My Dear Aunt Sally." PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. It is important to understand that multiplication and division have equal precedence, as do addition and subtraction, so those pairs of operations are simply worked from left to right in order.

> ➤ **Review Video: <u>Order of Operations</u>**
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Example: Evaluate the expression $5 + 20 \div 4 \times (2 + 3)^2 - 6$ using the correct order of operations.
P: Perform the operations inside the parentheses, $(2 + 3) = 5$.
E: Simplify the exponents, $(5)^2 = 25$.
The equation now looks like this: $5 + 20 \div 4 \times 25 - 6$.
MD: Perform multiplication and division from left to right, $20 \div 4 = 5$; then $5 \times 25 = 125$.
The equation now looks like this: $5 + 125 - 6$.
AS: Perform addition and subtraction from left to right, $5 + 125 = 130$; then $130 - 6 = 124$.

The laws of exponents are as follows:
1) Any number to the power of 1 is equal to itself: $a^1 = a$.
2) The number 1 raised to any power is equal to 1: $1^n = 1$.
3) Any number raised to the power of 0 is equal to 1: $a^0 = 1$.
4) Add exponents to multiply powers of the same base number: $a^n \times a^m = a^{n+m}$.
5) Subtract exponents to divide powers of the same number; that is $a^n \div a^m = a^{n-m}$.
6) Multiply exponents to raise a power to a power: $(a^n)^m = a^{n \times m}$.

7) If multiplied or divided numbers inside parentheses are collectively raised to a power, this is the same as each individual term being raised to that power: $(a \times b)^n = a^n \times b^n$; $(a \div b)^n = a^n \div b^n$.

Note: Exponents do not have to be integers. Fractional or decimal exponents follow all the rules above as well. Example: $5^{\frac{1}{4}} \times 5^{\frac{3}{4}} = 5^{\frac{1}{4}+\frac{3}{4}} = 5^1 = 5$.

A root, such as a square root, is another way of writing a fractional exponent. Instead of using a superscript, roots use the radical symbol ($\sqrt{\ }$) to indicate the operation. A radical will have a number underneath the bar, and may sometimes have a number in the upper left: $\sqrt[n]{a}$, read as "the n^{th} root of a." The relationship between radical notation and exponent notation can be described by this equation: $\sqrt[n]{a} = a^{\frac{1}{n}}$. The two special cases of $n = 2$ and $n = 3$ are called square roots and cube roots. If there is no number to the upper left, it is understood to be a square root ($n = 2$). Nearly all of the roots you encounter will be square roots. A square root is the same as a number raised to the one-half power. When we say that a is the square root of b ($a = \sqrt{b}$), we mean that a multiplied by itself equals b: ($a \times a = b$).

> **Review Video: Square Root and Perfect Square**
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A perfect square is a number that has an integer for its square root. There are 10 perfect squares from 1 to 100: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 (the squares of integers 1 through 10). Parentheses are used to designate which operations should be done first when there are multiple operations. Example: 4 – (2 + 1) = 1; the parentheses tell us that we must add 2 and 1, and then subtract the sum from 4, rather than subtracting 2 from 4 and then adding 1 (this would give us an answer of 3).

Scientific Notation

Scientific notation is a way of writing large numbers in a shorter form. The form $a \times 10^n$ is used in scientific notation, where a is greater than or equal to 1, but less than 10, and n is the number of places the decimal must move to get from the original number to a. Example: The number 230,400,000 is cumbersome to write. To write the value in scientific notation, place a decimal point between the first and second numbers, and include all digits through the last non-zero digit ($a = 2.304$). To find the appropriate power of 10, count the number of places the decimal point had to move ($n = 8$). The number is positive if the decimal moved to the left, and negative if it moved to the right. We can then write 230,400,000 as 2.304×10^8. If we look instead at the number 0.00002304, we have the same value for a, but this time the decimal moved 5 places to the right ($n = -5$). Thus, 0.00002304 can be written as 2.304×10^{-5}. Using this notation makes it simple to compare very large or very small numbers. By comparing exponents, it is easy to see that 3.28×10^4 is smaller than 1.51×10^5, because 4 is less than 5.

> **Review Video: Scientific Notation**
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Fractions, percentages, and related concepts

A fraction is a number that is expressed as one integer written above another integer, with a dividing line between them $\left(\frac{x}{y}\right)$. It represents the quotient of the two numbers "x divided by y." It can also be thought of as x out of y equal parts.

The top number of a fraction is called the numerator, and it represents the number of parts under consideration. The 1 in $\frac{1}{4}$ means that 1 part out of the whole is being considered in the calculation. The bottom number of a fraction is called the denominator, and it represents the total number of equal parts. The 4 in $\frac{1}{4}$ means that the whole consists of 4 equal parts. A fraction cannot have a denominator of zero; this is referred to as "undefined."

> **Review Video: Fractions**
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Fractions can be manipulated, without changing the value of the fraction, by multiplying or dividing (but not adding or subtracting) both the numerator and denominator by the same number. If you divide both numbers by a common factor, you are reducing or simplifying the fraction. Two fractions that have the same value, but are expressed differently are known as equivalent fractions. For example, $\frac{2}{10}, \frac{3}{15}, \frac{4}{20}$, and $\frac{5}{25}$ are all equivalent fractions. They can also all be reduced or simplified to $\frac{1}{5}$.

When two fractions are manipulated so that they have the same denominator, this is known as finding a common denominator. The number chosen to be that common denominator should be the least common multiple of the two original denominators. Example: $\frac{3}{4}$ and $\frac{5}{6}$; the least common multiple of 4 and 6 is 12. Manipulating to achieve the common denominator: $\frac{3}{4} = \frac{9}{12}; \frac{5}{6} = \frac{10}{12}$.

If two fractions have a common denominator, they can be added or subtracted simply by adding or subtracting the two numerators and retaining the same denominator. Example: $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. If the two fractions do not already have the same denominator, one or both of them must be manipulated to achieve a common denominator before they can be added or subtracted.

Two fractions can be multiplied by multiplying the two numerators to find the new numerator and the two denominators to find the new denominator. Example: $\frac{1}{3} \times \frac{2}{3} = \frac{1 \times 2}{3 \times 3} = \frac{2}{9}$.

> **Review Video: Multiplying Fractions**
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Two fractions can be divided flipping the numerator and denominator of the second fraction and then proceeding as though it were a multiplication. Example: $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$.

> **Review Video: Dividing Fractions**
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A fraction whose denominator is greater than its numerator is known as a proper fraction, while a fraction whose numerator is greater than its denominator is known as an improper fraction. Proper fractions have values less than one and improper fractions have values greater than one.

A mixed number is a number that contains both an integer and a fraction. Any improper fraction can be rewritten as a mixed number. Example: $\frac{8}{3} = \frac{6}{3} + \frac{2}{3} = 2 + \frac{2}{3} = 2\frac{2}{3}$. Similarly, any mixed number can be rewritten as an improper fraction. Example: $1\frac{3}{5} = 1 + \frac{3}{5} = \frac{5}{5} + \frac{3}{5} = \frac{8}{5}$.

> **Review Video: Improper Fractions and Mixed Numbers**
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Percentages can be thought of as fractions that are based on a whole of 100; that is, one whole is equal to 100%. The word percent means "per hundred." Fractions can be expressed as percents by finding equivalent fractions with a denomination of 100. Example: $\frac{7}{10} = \frac{70}{100} = 70\%$; $\frac{1}{4} = \frac{25}{100} = 25\%$.

To express a percentage as a fraction, divide the percentage number by 100 and reduce the fraction to its simplest possible terms. Example: $60\% = \frac{60}{100} = \frac{3}{5}$; $96\% = \frac{96}{100} = \frac{24}{25}$.

Converting decimals to percentages and percentages to decimals is as simple as moving the decimal point. To convert from a decimal to a percent, move the decimal point two places to the right. To convert from a percent to a decimal, move it two places to the left. Example: 0.23 = 23%; 5.34 = 534%; 0.007 = 0.7%; 700% = 7.00; 86% = 0.86; 0.15% = 0.0015.

It may be helpful to remember that the percentage number will always be larger than the equivalent decimal number.

A percentage problem can be presented three main ways: (1) Find what percentage of some number another number is. Example: What percentage of 40 is 8? (2) Find what number is some percentage of a given number. Example: What number is 20% of 40? (3) Find what number another number is a given percentage of. Example: What number is 8 20% of? The three components in all of these cases are the same: a whole (W), a part (P), and a percentage (%). These are related by the equation: $P = W \times \%$. This is the form of the equation you would use to solve problems of type (2). To solve types (1) and (3), you would use these two forms: $\% = \frac{P}{W}$ and $W = \frac{P}{\%}$.

> **Review Video: Percentages**
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The thing that frequently makes percentage problems difficult is that they are most often also word problems, so a large part of solving them is figuring out which quantities are what. Example: In a school cafeteria, 7 students choose pizza, 9 choose hamburgers, and 4 choose tacos. Find the percentage that chooses tacos. To find the whole, you must first add all of the parts: 7 + 9 + 4 = 20. The percentage can then be found by dividing the part by the whole $\left(\% = \frac{P}{W}\right)$: $\frac{4}{20} = \frac{20}{100} = 20\%$.

A ratio is a comparison of two quantities in a particular order. Example: If there are 14 computers in a lab, and the class has 20 students, there is a student to computer ratio of 20 to 14, commonly written as 20:14. Ratios are normally reduced to their smallest whole number representation, so 20:14 would be reduced to 10:7 by dividing both sides by 2.

A proportion is a relationship between two quantities that dictates how one changes when the other changes. A direct proportion describes a relationship in which a quantity increases by a set amount for every increase in the other quantity, or decreases by that same amount for every decrease in the other quantity. Example: Assuming a constant driving speed, the time required for a car trip increases as the distance of the trip increases. The distance to be traveled and the time required to travel are directly proportional.

Inverse proportion is a relationship in which an increase in one quantity is accompanied by a decrease in the other, or vice versa. Example: the time required for a car trip decreases as the speed increases, and increases as the speed decreases, so the time required is inversely proportional to the speed of the car.

Measurement, Geometry, and Data

Measurement equivalents

<u>1 yard in feet and inches</u>
1 yard = 3 feet
1 yard = 36 inches

<u>1 mile in feet and yards</u>
1 mile = 5280 feet
1 mile = 1760 yards

<u>1 acre in square feet</u>
1 acre = 43,560 square feet

<u>1 quart in pints and cups</u>
1 quart = 2 pints
1 quart = 4 cups

<u>1 gallon in quarts, pints, and cups</u>
1 gallon = 4 quarts
1 gallon = 8 pints
1 gallon = 16 cups

<u>1 pound in ounces</u>
1 pound = 16 ounces

Do not assume that because something weighs one pound that its volume is one pint. Ounces of weight are not equivalent to fluid ounces, which measure volume.

<u>1 ton in pounds</u>
1 ton = 2000 pounds

In the United States, the word "ton" by itself refers to a short ton or a net ton. Do not confuse this with a long ton (also called a gross ton) or a metric ton (also spelled *tonne*), which have different measurement equivalents.

Fluid measurements

1 cup in fluid ounces
1 cup = 8 fluid ounces

Note: This does NOT mean that one cup of something is the same as a half pound. Fluid ounces are measures of volume and have no correspondence with measures of weight.

1 pint in cups and fluid ounces
1 pint = 2 cups
1 pint = 16 ounces

Again, the phrase, "A pint's a pound the world round," does not apply. A pint of something does not necessarily weigh one pound, since one fluid ounce is not the same as one ounce in weight. The expression is valid only for helping you remember the number 16, since most people can remember there are 16 ounces in a pound.

Metric measurements

1 liter in milliliters and cubic centimeters
1 liter = 1000 milliliters
1 liter = 1000 cubic centimeters

Do not confuse *cubic centimeters* with *centiliters*. 1 liter = 1000 cubic centi*meters*, but 1 liter = 100 centi*liters*.

1 meter in millimeters and centimeters
1 meter = 1000 millimeters
1 meter = 100 centimeters

1 gram in milligrams
1 gram = 1000 milligrams

1 kilogram in grams
1 kilogram = 1000 grams

Kilo, centi, and milli

Kilo-: one thousand (1 *kilo*gram is one thousand grams.)
Centi-: one hundredth (1 *centi*meter is one hundredth of a meter.)
Milli-: one thousandth (1 *milli*liter is one thousandth of a liter.)

Converting unit measurements

When going from a larger unit to a smaller unit, multiply the numerical value of the known amount by the equivalent amount. When going from a smaller unit to a larger unit, divide the numerical value of the known amount by the equivalent amount. Also, you can set up conversion fractions where one fraction is the conversion fact, with the unit of the unknown amount in the numerator and the unit of the known value in the denominator. The second fraction has the known value from the problem in the numerator, and the unknown in the denominator. Multiply the two fractions to get the converted measurement.

Precision and accuracy

Precision: How reliable and repeatable a measurement is. The more consistent the data is with repeated testing, the more precise it is. For example, hitting a target consistently in the same spot, which may or may not be the center of the target, is precision.

Accuracy: How close the data is to the correct data. For example, hitting a target consistently in the center area of the target, whether or not the hits are all in the same spot, is accuracy.

Note that it is possible for data to be precise without being accurate. If a scale is off balance, the data will be precise, but will not be accurate. For data to have precision and accuracy, it must be repeatable and correct.

Approximate and maximum possible error

Approximate Error: The amount of error in a physical measurement. Approximate error is often reported as the measurement, followed by the ± symbol and the amount of the approximate error.

Maximum Possible Error: Half the magnitude of the smallest unit used in the measurement. For example, if the unit of measurement is 1 centimeter, the maximum possible error is $\frac{1}{2}$ cm, written as ± 0.5 cm following the measurement. It is important to apply significant figures in reporting maximum possible error. Do not make the answer appear more accurate than the least accurate of your measurements.

Lines and planes

A point is a fixed location in space; has no size or dimensions; commonly represented by a dot.

A line is a set of points that extends infinitely in two opposite directions. It has length, but no width or depth. A line can be defined by any two distinct points that it contains. A line segment is a portion of a line that has definite endpoints. A ray is a portion of a line that extends from a single point on that line in one direction along the line. It has a definite beginning, but no ending.

A plane is a two-dimensional flat surface defined by three non-collinear points. A plane extends an infinite distance in all directions in those two dimensions. It contains an infinite number of points, parallel lines and segments, intersecting lines and segments, as well as parallel or intersecting rays. A plane will never contain a three-dimensional figure or skew lines. Two given planes will either be parallel or they will intersect to form a line. A plane may intersect a circular conic surface, such as a cone, to form conic sections, such as the parabola, hyperbola, circle or ellipse.

Perpendicular lines are lines that intersect at right angles. They are represented by the symbol ⊥. The shortest distance from a line to a point not on the line is a perpendicular segment from the point to the line.

Parallel lines are lines in the same plane that have no points in common and never meet. It is possible for lines to be in different planes, have no points in common, and never meet, but they are not parallel because they are in different planes.

A bisector is a line or line segment that divides another line segment into two equal lengths. A perpendicular bisector of a line segment is composed of points that are equidistant from the endpoints of the segment it is dividing.

Intersecting lines are lines that have exactly one point in common. Concurrent lines are multiple lines that intersect at a single point.

A transversal is a line that intersects at least two other lines, which may or may not be parallel to one another. A transversal that intersects parallel lines is a common occurrence in geometry.

Angles

An angle is formed when two lines or line segments meet at a common point. It may be a common starting point for a pair of segments or rays, or it may be the intersection of lines. Angles are represented by the symbol ∠.

The vertex is the point at which two segments or rays meet to form an angle. If the angle is formed by intersecting rays, lines, and/or line segments, the vertex is the point at which four angles are formed. The pairs of angles opposite one another are called vertical angles, and their measures are equal.

An acute angle is an angle with a degree measure less than 90°.
A right angle is an angle with a degree measure of exactly 90°.
An obtuse angle is an angle with a degree measure greater than 90° but less than 180°.
A straight angle is an angle with a degree measure of exactly 180°. This is also a semicircle.
A reflex angle is an angle with a degree measure greater than 180° but less than 360°.
A full angle is an angle with a degree measure of exactly 360°.

➤ **Review Video: Geometric Symbols: Angles**
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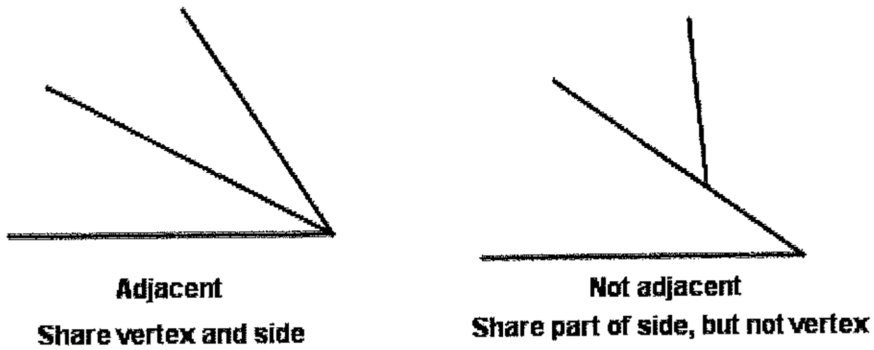
Two angles whose sum is exactly 90° are said to be complementary. The two angles may or may not be adjacent. In a right triangle, the two acute angles are complementary.

Two angles whose sum is exactly 180° are said to be supplementary. The two angles may or may not be adjacent. Two intersecting lines always form two pairs of supplementary angles. Adjacent supplementary angles will always form a straight line.

Two angles that have the same vertex and share a side are said to be adjacent. Vertical angles are not adjacent because they share a vertex but no common side.



When two parallel lines are cut by a transversal, the angles that are between the two parallel lines are interior angles. In the diagram below, angles 3, 4, 5, and 6 are interior angles.

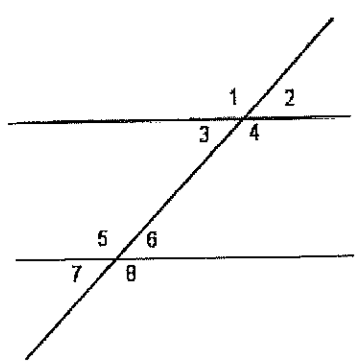
When two parallel lines are cut by a transversal, the angles that are outside the parallel lines are exterior angles. In the diagram below, angles 1, 2, 7, and 8 are exterior angles.

When two parallel lines are cut by a transversal, the angles that are in the same position relative to the transversal and a parallel line are corresponding angles. The diagram below has four pairs of corresponding angles: angles 1 and 5; angles 2 and 6; angles 3 and 7; and angles 4 and 8. Corresponding angles formed by parallel lines are congruent.

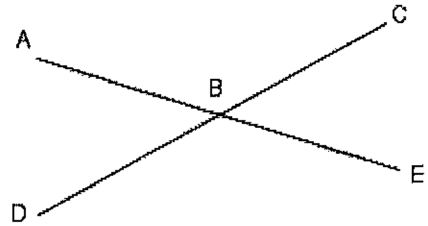
When two parallel lines are cut by a transversal, the two interior angles that are on opposite sides of the transversal are called alternate interior angles. In the diagram below, there are two pairs of alternate interior angles: angles 3 and 6, and angles 4 and 5. Alternate interior angles formed by parallel lines are congruent.

When two parallel lines are cut by a transversal, the two exterior angles that are on opposite sides of the transversal are called alternate exterior angles.

In the diagram below, there are two pairs of alternate exterior angles: angles 1 and 8, and angles 2 and 7. Alternate exterior angles formed by parallel lines are congruent.

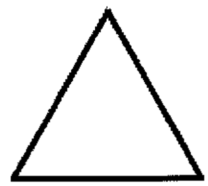


When two lines intersect, four angles are formed. The non-adjacent angles at this vertex are called vertical angles. Vertical angles are congruent. In the diagram, $\angle ABD \cong \angle CBE$ and $\angle ABC \cong \angle DBE$.

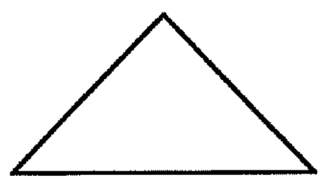


Triangles

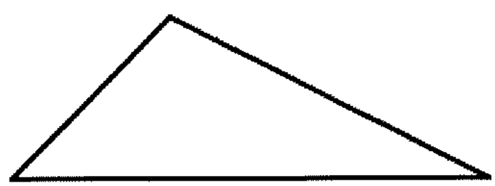
An equilateral triangle is a triangle with three congruent sides. An equilateral triangle will also have three congruent angles, each 60°. All equilateral triangles are also acute triangles.



An isosceles triangle is a triangle with two congruent sides. An isosceles triangle will also have two congruent angles opposite the two congruent sides.



A scalene triangle is a triangle with no congruent sides. A scalene triangle will also have three angles of different measures. The angle with the largest measure is opposite the longest side, and the angle with the smallest measure is opposite the shortest side.



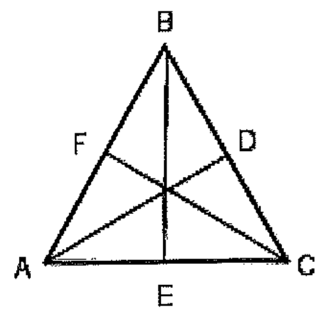
An acute triangle is a triangle whose three angles are all less than 90°. If two of the angles are equal, the acute triangle is also an isosceles triangle. If the three angles are all equal, the acute triangle is also an equilateral triangle.

A right triangle is a triangle with exactly one angle equal to 90°. All right triangles follow the Pythagorean Theorem. A right triangle can never be acute or obtuse.

An obtuse triangle is a triangle with exactly one angle greater than 90°. The other two angles may or may not be equal. If the two remaining angles are equal, the obtuse triangle is also an isosceles triangle.

Terminology

Altitude of a Triangle: A line segment drawn from one vertex perpendicular to the opposite side. In the diagram below, $\overline{BE}$, $\overline{AD}$, and $\overline{CF}$ are altitudes. The three altitudes in a triangle are always concurrent.



Height of a Triangle: The length of the altitude, although the two terms are often used interchangeably.

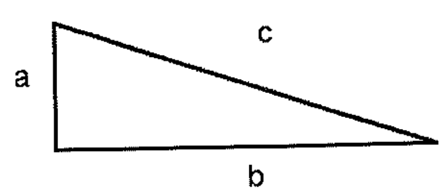
Orthocenter of a Triangle: The point of concurrency of the altitudes of a triangle. Note that in an obtuse triangle, the orthocenter will be outside the circle, and in a right triangle, the orthocenter is the vertex of the right angle.

Median of a Triangle: A line segment drawn from one vertex to the midpoint of the opposite side. This is not the same as the altitude, except the altitude to the base of an isosceles triangle and all three altitudes of an equilateral triangle.

Centroid of a Triangle: The point of concurrency of the medians of a triangle. This is the same point as the orthocenter only in an equilateral triangle. Unlike the orthocenter, the centroid is always inside the triangle. The centroid can also be considered the exact center of the triangle. Any shape triangle can be perfectly balanced on a tip placed at the centroid. The centroid is also the point that is two-thirds the distance from the vertex to the opposite side.

Pythagorean Theorem

The side of a triangle opposite the right angle is called the hypotenuse. The other two sides are called the legs. The Pythagorean Theorem states a relationship among the legs and hypotenuse of a right triangle: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse. Note that this formula will only work with right triangles.



General rules

The Triangle Inequality Theorem states that the sum of the measures of any two sides of a triangle is always greater than the measure of the third side. If the sum of the measures of two sides were equal to the third side, a triangle would be impossible because the two sides would lie flat across the third side and there would be no vertex. If the sum of the measures of two of the sides was less than the third side, a closed figure would be impossible because the two shortest sides would never meet.

The sum of the measures of the interior angles of a triangle is always 180°. Therefore, a triangle can never have more than one angle greater than or equal to 90°.

In any triangle, the angles opposite congruent sides are congruent, and the sides opposite congruent angles are congruent. The largest angle is always opposite the longest side, and the smallest angle is always opposite the shortest side.

The line segment that joins the midpoints of any two sides of a triangle is always parallel to the third side and exactly half the length of the third side.

Similarity and congruence rules

Similar triangles are triangles whose corresponding angles are equal and whose corresponding sides are proportional. Represented by AA. Similar triangles whose corresponding sides are congruent are also congruent triangles.

> **Review Video:** Similar Triangles
> *Visit* ***mometrix.com/academy*** *and enter* ***Code*****: 398538**

Three sides of one triangle are congruent to the three corresponding sides of the second triangle. Represented as SSS.

Two sides and the included angle (the angle formed by those two sides) of one triangle are congruent to the corresponding two sides and included angle of the second triangle. Represented by SAS.

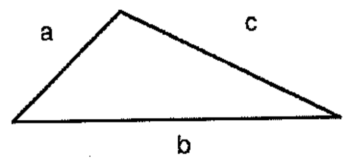
Two angles and the included side (the side that joins the two angles) of one triangle are congruent to the corresponding two angles and included side of the second triangle. Represented by ASA.

Two angles and a non-included side of one triangle are congruent to the corresponding two angles and non-included side of the second triangle. Represented by AAS.

Note that AAA is not a form for congruent triangles. This would say that the three angles are congruent, but says nothing about the sides. This meets the requirements for similar triangles, but not congruent triangles.

Area and perimeter formulas

The perimeter of any triangle is found by summing the three side lengths; $P = a + b + c$. For an equilateral triangle, this is the same as $P = 3s$, where s is any side length, since all three sides are the same length.

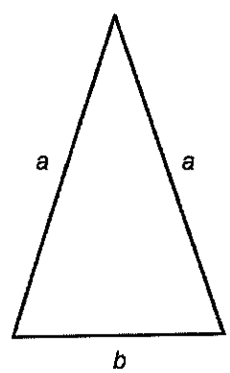


The area of any triangle can be found by taking half the product of one side length (base or b) and the perpendicular distance from that side to the opposite vertex (height or h). In equation form, $A = \frac{1}{2}bh$. For many triangles, it may be difficult to calculate h, so using one of the other formulas given here may be easier.

Another formula that works for any triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where A is the area, s is the semiperimeter $s = \frac{a+b+c}{2}$, and a, b, and c are the lengths of the three sides.

The area of an equilateral triangle can be found by the formula $A = \frac{\sqrt{3}}{4}s^2$, where A is the area and s is the length of a side. You could use the $30° - 60° - 90°$ ratios to find the height of the triangle and then use the standard triangle area formula, but this is faster.

The area of an isosceles triangle can be found by the formula, $A = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$, where A is the area, b is the base (the unique side), and a is the length of one of the two congruent sides. If you do not remember this formula, you can use the Pythagorean Theorem to find the height so you can use the standard formula for the area of a triangle.



> **Review Video: <u>Area and Perimeter of a Triangle</u>**
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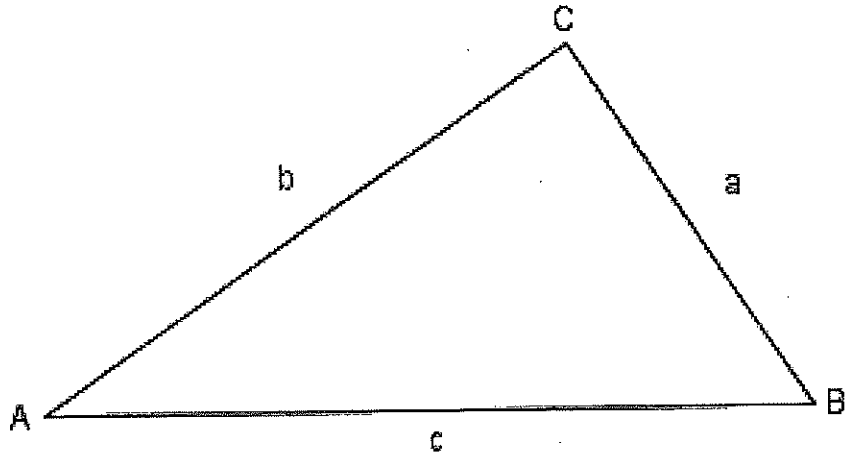
Trigonometric formulas

In the diagram below, angle C is the right angle, and side c is the hypotenuse. Side a is the side adjacent to angle B and side b is the side adjacent to angle A. These formulas will work for any acute angle in a right triangle. They will NOT work for any triangle that is not a right triangle. Also, they will not work for the right angle in a right triangle, since there are not distinct adjacent and opposite sides to differentiate from the hypotenuse.



$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$

$$\csc A = \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$$

$$\sec A = \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

$$\cot A = \frac{1}{\tan A} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$$

Laws of Sines and Cosines

The Law of Sines states that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where A, B, and C are the angles of a triangle, and a, b, and c are the sides opposite their respective angles. This formula will work with all triangles, not just right triangles.

The Law of Cosines is given by the formula $c^2 = a^2 + b^2 - 2ab(\cos C)$, where a, b, and c are the sides of a triangle, and C is the angle opposite side c. This formula is similar to the Pythagorean Theorem, but unlike the Pythagorean Theorem, it can be used on any triangle.

➤ **Review Video: Cosine**
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Polygons

Each straight line segment of a polygon is called a side.

The point at which two sides of a polygon intersect is called the vertex. In a polygon, the number of sides is always equal to the number of vertices.

A polygon with all sides congruent and all angles equal is called a regular polygon.

A line segment from the center of a polygon perpendicular to a side of the polygon is called the apothem. In a regular polygon, the apothem can be used to find the area of the polygon using the formula $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.

A line segment from the center of a polygon to a vertex of the polygon is called a radius. The radius of a regular polygon is also the radius of a circle that can be circumscribed about the polygon.

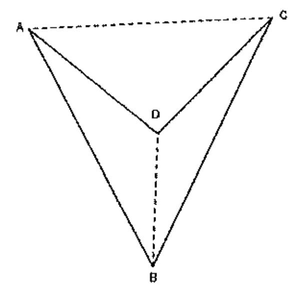
Triangle – 3 sides
Quadrilateral – 4 sides
Pentagon – 5 sides
Hexagon – 6 sides
Heptagon – 7 sides
Octagon – 8 sides
Nonagon – 9 sides
Decagon – 10 sides
Dodecagon – 12 sides

More generally, an n-gon is a polygon that has n angles and n sides.

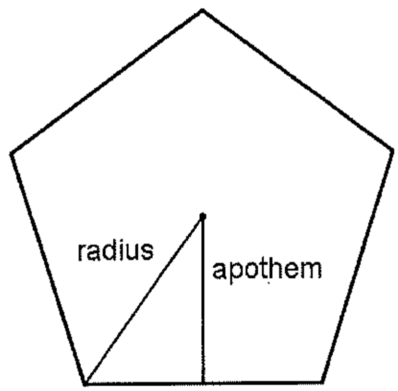
The sum of the interior angles of an n-sided polygon is $(n - 2)180°$. For example, in a triangle $n = 3$, so the sum of the interior angles is $(3 - 2)180° = 180°$. In a quadrilateral, $n = 4$, and the sum of the angles is $(4 - 2)180° = 360°$. The sum of the interior angles of a polygon is equal to the sum of the interior angles of any other polygon with the same number of sides.

A diagonal is a line segment that joins two non-adjacent vertices of a polygon.
A convex polygon is a polygon whose diagonals all lie within the interior of the polygon.

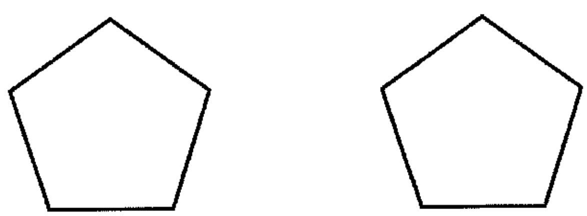
A concave polygon is a polygon with a least one diagonal that lies outside the polygon. In the diagram below, quadrilateral $ABCD$ is concave because diagonal $\overline{AC}$ lies outside the polygon.



The number of diagonals a polygon has can be found by using the formula: number of diagonals $= \frac{n(n-3)}{2}$, where n is the number of sides in the polygon. This formula works for all polygons, not just regular polygons.

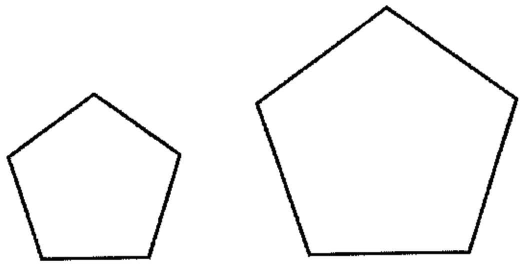


Congruent figures are geometric figures that have the same size and shape. All corresponding angles are equal, and all corresponding sides are equal. It is indicated by the symbol ≅.



Congruent polygons

Similar figures are geometric figures that have the same shape, but do not necessarily have the same size. All corresponding angles are equal, and all corresponding sides are proportional, but they do not have to be equal. It is indicated by the symbol ~.



Similar polygons

Note that all congruent figures are also similar, but not all similar figures are congruent.

> **Review Video: Polygons, Similarity, and Congruence**
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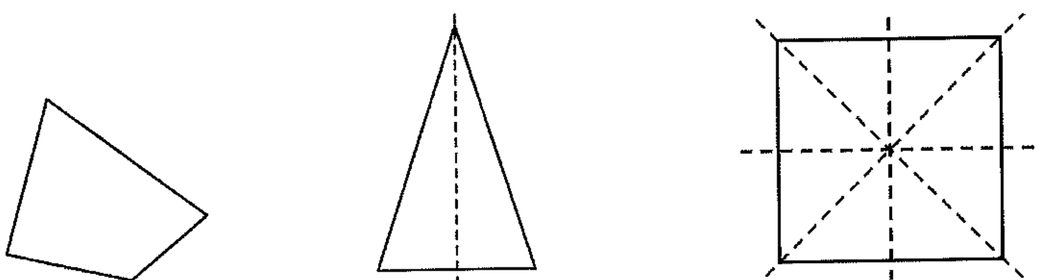
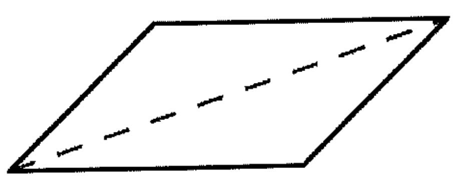
Line of Symmetry: The line that divides a figure or object into two symmetric parts. Each symmetric half is congruent to the other. An object may have no lines of symmetry, one line of symmetry, or more than one line of symmetry.



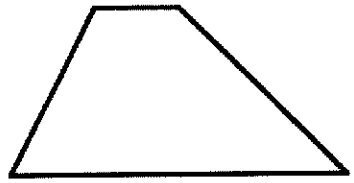
No lines of symmetry One line of symmetry Multiple lines of symmetry

Quadrilateral: A closed two-dimensional geometric figure composed of exactly four straight sides. The sum of the interior angles of any quadrilateral is 360°.

Parallelogram: A quadrilateral that has exactly two pairs of opposite parallel sides. The sides that are parallel are also congruent. The opposite interior angles are always congruent, and the consecutive interior angles are supplementary. The diagonals of a parallelogram bisect each other. Each diagonal divides the parallelogram into two congruent triangles.

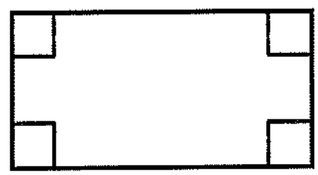


Trapezoid: Traditionally, a quadrilateral that has exactly one pair of parallel sides. Some math texts define trapezoid as a quadrilateral that has at least one pair of parallel sides. Because there are no rules governing the second pair of sides, there are no rules that apply to the properties of the diagonals of a trapezoid.

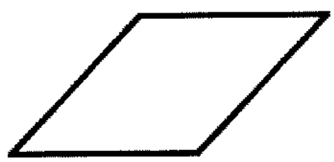


Rectangles, rhombuses, and squares are all special forms of parallelograms.

Rectangle: A parallelogram with four right angles. All rectangles are parallelograms, but not all parallelograms are rectangles. The diagonals of a rectangle are congruent.



Rhombus: A parallelogram with four congruent sides. All rhombuses are parallelograms, but not all parallelograms are rhombuses. The diagonals of a rhombus are perpendicular to each other.



- **Review Video: <u>Diagonals of Parallelograms, Rectangles, and Rhombi</u>**
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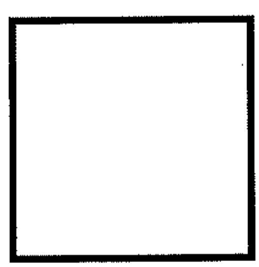
Square: A parallelogram with four right angles and four congruent sides. All squares are also parallelograms, rhombuses, and rectangles. The diagonals of a square are congruent and perpendicular to each other.



A quadrilateral whose diagonals bisect each other is a parallelogram. A quadrilateral whose opposite sides are parallel (2 pairs of parallel sides) is a parallelogram.

A quadrilateral whose diagonals are perpendicular bisectors of each other is a rhombus. A quadrilateral whose opposite sides (both pairs) are parallel and congruent is a rhombus.

A parallelogram that has a right angle is a rectangle. (Consecutive angles of a parallelogram are supplementary. Therefore if there is one right angle in a parallelogram, there are four right angles in that parallelogram.)

A rhombus with one right angle is a square. Because the rhombus is a special form of a parallelogram, the rules about the angles of a parallelogram also apply to the rhombus.

Area and perimeter formulas

The area of a square is found by using the formula $A = s^2$, where and s is the length of one side.

The perimeter of a square is found by using the formula $P = 4s$, where s is the length of one side. Because all four sides are equal in a square, it is faster to multiply the length of one side by 4 than to add the same number four times. You could use the formulas for rectangles and get the same answer.

> **Review Video: <u>Area and Perimeter of a Square</u>**
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The area of a rectangle is found by the formula $A = lw$, where A is the area of the rectangle, l is the length (usually considered to be the longer side) and w is the width (usually considered to be the shorter side). The numbers for l and w are interchangeable.

The perimeter of a rectangle is found by the formula $P = 2l + 2w$ or $P = 2(l + w)$, where l is the length, and w is the width. It may be easier to add the length and width first and then double the result, as in the second formula.

> **Review Video: <u>Area and Perimeter of a Rectangle</u>**
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The area of a parallelogram is found by the formula $A = bh$, where b is the length of the base, and h is the height. Note that the base and height correspond to the length and width in a rectangle, so this formula would apply to rectangles as well. Do not confuse the height of a parallelogram with the length of the second side. The two are only the same measure in the case of a rectangle.

The perimeter of a parallelogram is found by the formula $P = 2a + 2b$ or $P = 2(a + b)$, where a and b are the lengths of the two sides.

> **Review Video: <u>Area and Perimeter of a Parallelogram</u>**
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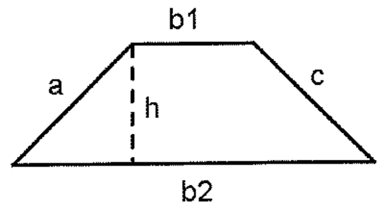
The area of a trapezoid is found by the formula $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height (segment joining and perpendicular to the parallel bases), and b_1 and b_2 are the two parallel sides (bases). Do not use one of the other two sides as the height unless that side is also perpendicular to the parallel bases.

The perimeter of a trapezoid is found by the formula $P = a + b_1 + c + b_2$, where a, b_1, c, and b_2 are the four sides of the trapezoid.



➤ **Review Video: <u>Area and Perimeter of a Trapezoid</u>**
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Circles

The center is the single point inside the circle that is equidistant from every point on the circle. (Point *O* in the diagram below.)

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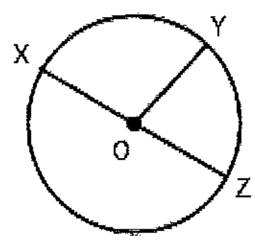
The radius is a line segment that joins the center of the circle and any one point on the circle. All radii of a circle are equal. (Segments *OX*, *OY*, and *OZ* in the diagram below.)

The diameter is a line segment that passes through the center of the circle and has both endpoints on the circle. The length of the diameter is exactly twice the length of the radius. (Segment *XZ* in the diagram below.)



The area of a circle is found by the formula $A = \pi r^2$, where *r* is the length of the radius. If the diameter of the circle is given, remember to divide it in half to get the length of the radius before proceeding.

The circumference of a circle is found by the formula $C = 2\pi r$, where *r* is the radius. Again, remember to convert the diameter if you are given that measure rather than the radius.

➤ **Review Video: <u>Area and Circumference of a Circle</u>**
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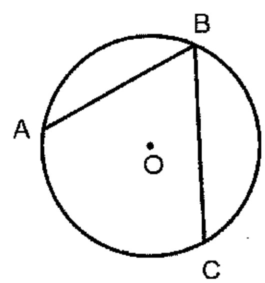
Concentric circles are circles that have the same center, but not the same length of radii. A bulls-eye target is an example of concentric circles.

An arc is a portion of a circle. Specifically, an arc is the set of points between and including two points on a circle. An arc does not contain any points inside the circle. When a segment is drawn from the endpoints of an arc to the center of the circle, a sector is formed.

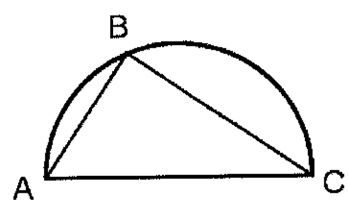
A central angle is an angle whose vertex is the center of a circle and whose legs intercept an arc of the circle. Angle *XOY* in the diagram above is a central angle. A minor arc is an arc that has a measure less than 180°. The measure of a central angle is equal to the measure of the minor arc it intercepts. A major arc is an arc having a measure of at least 180°. The measure of the major arc can be found by subtracting the measure of the central angle from 360°.

A semicircle is an arc whose endpoints are the endpoints of the diameter of a circle. A semicircle is exactly half of a circle.

An inscribed angle is an angle whose vertex lies on a circle and whose legs contain chords of that circle. The portion of the circle intercepted by the legs of the angle is called the intercepted arc. The measure of the intercepted arc is exactly twice the measure of the inscribed angle. In the following diagram, angle *ABC* is an inscribed angle. $\widehat{AC} = 2(m\angle ABC)$

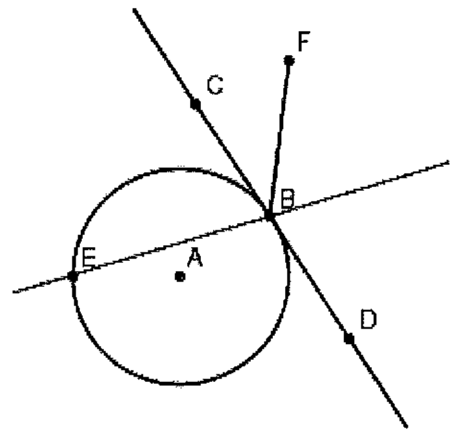


Any angle inscribed in a semicircle is a right angle. The intercepted arc is 180°, making the inscribed angle half that, or 90°. In the diagram below, angle *ABC* is inscribed in semicircle *ABC*, making angle *ABC* equal to 90°.



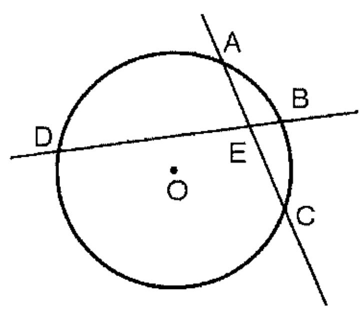
A chord is a line segment that has both endpoints on a circle. In the diagram below, $\overline{EB}$ is a chord. Secant: A line that passes through a circle and contains a chord of that circle. In the diagram below, $\overleftrightarrow{EB}$ is a secant and contains chord $\overline{EB}$.

A tangent is a line in the same plane as a circle that touches the circle in exactly one point. While a line segment can be tangent to a circle as part of a line that is tangent, it is improper to say a tangent can be simply a line segment that touches the circle in exactly one point. In the diagram below, $\overleftrightarrow{CD}$ is tangent to circle A. Notice that $\overline{FB}$ is not tangent to the circle. $\overline{FB}$ is a line segment that touches the circle in exactly one point, but if the segment were extended, it would touch the circle in a second point. The point at which a tangent touches a circle is called the point of tangency. In the diagram below, point B is the point of tangency.

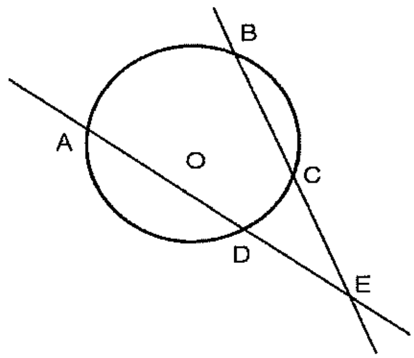


A secant is a line that intersects a circle in two points. Two secants may intersect inside the circle, on the circle, or outside the circle. When the two secants intersect on the circle, an inscribed angle is formed.

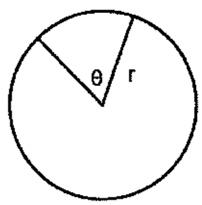
When two secants intersect inside a circle, the measure of each of two vertical angles is equal to half the sum of the two intercepted arcs. In the diagram below, $m\angle AEB = \frac{1}{2}(\widehat{AB} + \widehat{CD})$ and $m\angle BEC = \frac{1}{2}(\widehat{BC} + \widehat{AD})$.



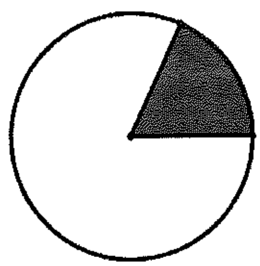
When two secants intersect outside a circle, the measure of the angle formed is equal to half the difference of the two arcs that lie between the two secants. In the diagram below, $m\angle E = \frac{1}{2}(\widehat{AB} - \widehat{CD})$.



The arc length is the length of that portion of the circumference between two points on the circle. The formula for arc length is $s = \frac{\pi r\theta}{180°}$ where s is the arc length, r is the length of the radius, and θ is the angular measure of the arc in degrees, or $s = r\theta$, where θ is the angular measure of the arc in radians (2π radians $= 360$ degrees).



A sector is the portion of a circle formed by two radii and their intercepted arc. While the arc length is exclusively the points that are also on the circumference of the circle, the sector is the entire area bounded by the arc and the two radii.

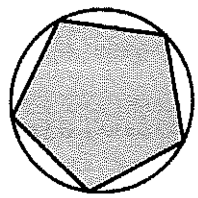


The area of a sector of a circle is found by the formula, $A = \frac{\theta r^2}{2}$, where A is the area, θ is the measure of the central angle in radians, and r is the radius. To find the area when the central angle is in degrees, use the formula, $A = \frac{\theta\pi r^2}{360}$, where θ is the measure of the central angle in degrees and r is the radius.

A circle is inscribed in a polygon if each of the sides of the polygon is tangent to the circle. A polygon is inscribed in a circle if each of the vertices of the polygon lies on the circle.

A circle is circumscribed about a polygon if each of the vertices of the polygon lies on the circle. A polygon is circumscribed about the circle if each of the sides of the polygon is tangent to the circle.

If one figure is inscribed in another, then the other figure is circumscribed about the first figure.



Circle circumscribed about a pentagon
Pentagon inscribed in a circle

Other conic sections

An ellipse is the set of all points in a plane, whose total distance from two fixed points called the foci (singular: focus) is constant, and whose center is the midpoint between the foci.

The standard equation of an ellipse that is taller than it is wide is $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$, where a and b are coefficients. The center is the point (h,k) and the foci are the points $(h,k+c)$ and $(h,k-c)$, where $c^2=a^2-b^2$ and $a^2>b^2$.

The major axis has length $2a$, and the minor axis has length $2b$.

Eccentricity (e) is a measure of how elongated an ellipse is, and is the ratio of the distance between the foci to the length of the major axis. Eccentricity will have a value between 0 and 1. The closer to 1 the eccentricity is, the closer the ellipse is to being a circle. The formula for eccentricity is $=\frac{c}{a}$.

Parabola: The set of all points in a plane that are equidistant from a fixed line, called the directrix, and a fixed point not on the line, called the focus.

Axis: The line perpendicular to the directrix that passes through the focus.

For parabolas that open up or down, the standard equation is $(x-h)^2=4c(y-k)$, where h, c, and k are coefficients. If c is positive, the parabola opens up. If c is negative, the parabola opens down. The vertex is the point (h,k). The directrix is the line having the equation $y=-c+k$, and the focus is the point $(h,c+k)$.

For parabolas that open left or right, the standard equation is $(y-k)^2=4c(x-h)$, where k, c, and h are coefficients. If c is positive, the parabola opens to the right. If c is negative, the parabola opens to the left. The vertex is the point (h,k). The directrix is the line having the equation $x=-c+h$, and the focus is the point $(c+h,k)$.

A hyperbola is the set of all points in a plane, whose distance from two fixed points, called foci, has a constant difference.

The standard equation of a horizontal hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where a, b, h, and k are real numbers. The center is the point (h, k), the vertices are the points $(h + a, k)$ and $(h - a, k)$, and the foci are the points that every point on one of the parabolic curves is equidistant from and are found using the formulas $(h + c, k)$ and $(h - c, k)$, where $c^2 = a^2 + b^2$. The asymptotes are two lines the graph of the hyperbola approaches but never reaches, and are given by the equations $y = \left(\frac{b}{a}\right)(x - h) + k$ and $y = -\left(\frac{b}{a}\right)(x - h) + k$.

A vertical hyperbola is formed when a plane makes a vertical cut through two cones that are stacked vertex-to-vertex.

The standard equation of a vertical hyperbola is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, where a, b, k, and h are real numbers. The center is the point (h, k), the vertices are the points $(h, k + a)$ and $(h, k - a)$, and the foci are the points that every point on one of the parabolic curves is equidistant from and are found using the formulas $(h, k + c)$ and $(h, k - c)$, where $c^2 = a^2 + b^2$. The asymptotes are two lines the graph of the hyperbola approaches but never reach, and are given by the equations $y = \left(\frac{a}{b}\right)(x - h) + k$ and $y = -\left(\frac{a}{b}\right)(x - h) + k$.

Solids

The surface area of a solid object is the area of all sides or exterior surfaces. For objects such as prisms and pyramids, a further distinction is made between base surface area (B) and lateral surface area (LA). For a prism, the total surface area (SA) is $SA = LA + 2B$. For a pyramid or cone, the total surface area is $SA = LA + B$.

➢ **Review Video: <u>Finding Volume in Geometry</u>**
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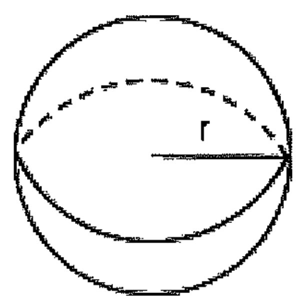
The surface area of a sphere can be found by the formula $A = 4\pi r^2$, where r is the radius. The volume is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. Both quantities are generally given in terms of π.



➢ **Review Video: <u>Volume and Surface Area of a Sphere</u>**
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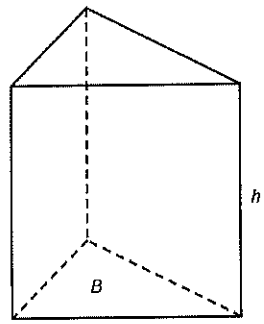
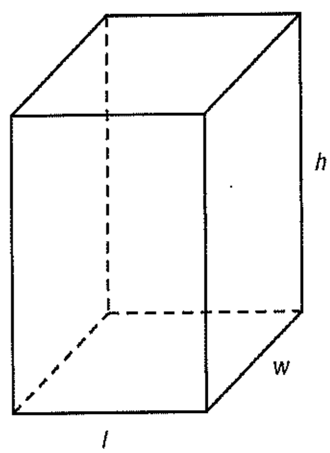
The volume of any prism is found by the formula $V = Bh$, where B is the area of the base, and h is the height (perpendicular distance between the bases). The surface area of any prism is the sum of the areas of both bases and all sides. It can be calculated as $SA = 2B + Ph$, where P is the perimeter of the base.



For a rectangular prism, the volume can be found by the formula $V = lwh$, where V is the volume, l is the length, w is the width, and h is the height. The surface area can be calculated as $SA = 2lw + 2hl + 2wh$ or $SA = 2(lw + hl + wh)$.



The volume of a cube can be found by the formula $V = s^3$, where s is the length of a side. The surface area of a cube is calculated as $SA = 6s^2$, where SA is the total surface area and s is the length of a side. These formulas are the same as the ones used for the volume and surface area of a rectangular prism, but simplified since all three quantities (length, width, and height) are the same.

➢ **Review Video: <u>Volume and Surface Area of a Cube</u>**
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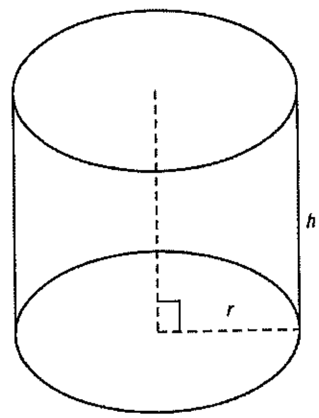
The volume of a cylinder can be calculated by the formula $V = \pi r^2 h$, where r is the radius, and h is the height. The surface area of a cylinder can be found by the formula $SA = 2\pi r^2 + 2\pi rh$. The first term is the base area multiplied by two, and the second term is the perimeter of the base multiplied by the height.



➢ **Review Video: Volume and Surface Area of a Right Circular Cylinder**
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The volume of a pyramid is found by the formula $V = \frac{1}{3}Bh$, where B is the area of the base, and h is the height (perpendicular distance from the vertex to the base). Notice this formula is the same as $\frac{1}{3}$ times the volume of a prism. Like a prism, the base of a pyramid can be any shape.

➢ **Review Video: Volume and Surface Area of a Pyramid**
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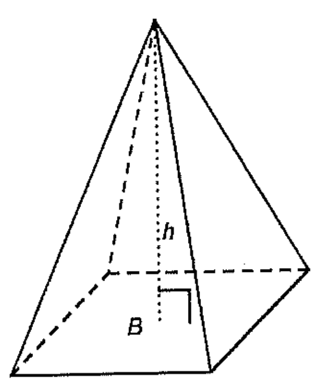
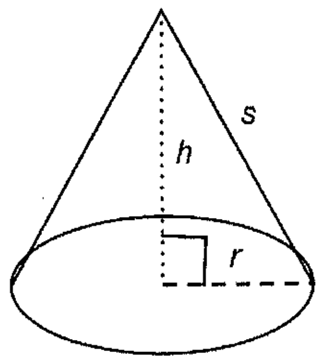
Finding the surface area of a pyramid is not as simple as the other shapes we've looked at thus far. If the pyramid is a right pyramid, meaning the base is a regular polygon and the vertex is directly over the center of that polygon, the surface area can be calculated as $SA = B + \frac{1}{2}Ph_s$, where P is the perimeter of the base, and h_s is the slant height (distance from the vertex to the midpoint of one side of the base). If the pyramid is irregular, the area of each triangle side must be calculated individually and then summed, along with the base.



The volume of a cone is found by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius, and h is the height. Notice this is the same as $\frac{1}{3}$ times the volume of a cylinder. The surface area can be calculated as $SA = \pi r^2 + \pi r s$, where s is the slant height. The slant height can be calculated using the Pythagorean Thereom to be $\sqrt{r^2 + h^2}$, so the surface area formula can also be written as $SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$.



➤ **Review Video: Volume and Surface Area of a Right Circular Cone**
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Statistics

In statistics, the *Population* is the entire collection of people, plants, etc., that data can be collected from. For example, a study to determine how well students in the area schools perform on a standardized test would have a population of all the students enrolled in those schools, although a study may include just a small sample of students from each school. A *Parameter* is a numerical value that gives information about the population, such as the mean, median, mode, or standard deviation. Remember that the symbol for the mean of a population is μ and the symbol for the standard deviation of a population is σ.

A *Sample* is a portion of the entire population. Where as a parameter helped describe the population, a *Statistic* is a numerical value that gives information about the sample, such as mean, median, mode, or standard deviation. Keep in mind that the symbols for mean and standard deviation are different when they are referring to a sample rather than the entire population. For a sample, the symbol for mean is $\bar{x}$ and the symbol for standard deviation is s. The mean and standard deviation of a sample may or may not be identical to that of the entire population due to a sample only being a subset of the population. However, if the sample is random and large enough, statistically significant values can be attained. Samples are generally used when the population is too large to justify including every element or when acquiring data for the entire population is impossible.

Inferential Statistics is the branch of statistics that uses samples to make predictions about an entire population. This type of statistics is often seen in political polls, where a sample of the population is questioned about a particular topic or politician to gain an understanding about the attitudes of the entire population of the country. Often, exit polls are conducted on election days using this method. Inferential statistics can have a large margin of error if you do not have a valid sample.

Statistical values calculated from various samples of the same size make up the sampling distribution. For example, if several samples of identical size are randomly selected from a large population and then the mean of each sample is calculated, the distribution of values of the means would be a *Sampling Distribution.*

The *Sampling Distribution of the Mean* is the distribution of the sample mean, $\bar{x}$, derived from random samples of a given size. It has three important characteristics. First, the mean of the sampling distribution of the mean is equal to the mean of the population that was sampled. Second, assuming the standard deviation is non-zero, the standard deviation of the sampling distribution of the mean equals the standard deviation of the sampled population divided by the square root of the sample size. This is sometimes called the standard error. Finally, as the sample size gets larger, the sampling distribution of the mean gets closer to a normal distribution via the Central Limit Theorem.

A *Survey Study* is a method of gathering information from a small group in an attempt to gain enough information to make accurate general assumptions about the population. Once a survey study is completed, the results are then put into a summary report.
Survey studies are generally in the format of surveys, interviews, or questionnaires as part of an effort to find opinions of a particular group or to find facts about a group.
It is important to note that the findings from a survey study are only as accurate as the sample chosen from the population.

Correlational Studies seek to determine how much one variable is affected by changes in a second variable. For example, correlational studies may look for a relationship between the amount of time a student spends studying for a test and the grade that student earned on the test or between student scores on college admissions tests and student grades in college.
It is important to note that correlational studies cannot show a cause and effect, but rather can show only that two variables are or are not potentially correlated.

Experimental Studies take correlational studies one step farther, in that they attempt to prove or disprove a cause-and-effect relationship. These studies are performed by conducting a series of experiments to test the hypothesis. For a study to be scientifically accurate, it must have both an experimental group that receives the specified treatment and a control group that does not get the treatment. This is the type of study pharmaceutical companies do as part of drug trials for new medications. Experimental studies are only valid when proper scientific method has been followed. In other words, the experiment must be well-planned and executed without bias in the testing process, all subjects must be selected at random, and the process of determining which subject is in which of the two groups must also be completely random.

Observational Studies are the opposite of experimental studies. In observational studies, the tester cannot change or in any way control all of the variables in the test. For example, a study to determine which gender does better in math classes in school is strictly observational. You cannot change a person's gender, and you cannot change the subject being studied. The big downfall of observational studies is that you have no way of proving a cause-and-effect relationship because you cannot control outside influences. Events outside of school can influence a student's performance in school, and observational studies cannot take that into consideration.

For most studies, a *Random Sample* is necessary to produce valid results. Random samples should not have any particular influence to cause sampled subjects to behave one way or another. The goal is for the random sample to be a *Representative Sample*, or a sample whose characteristics give an accurate picture of the characteristics of the entire population. To accomplish this, you must make sure you have a proper *Sample Size*, or an appropriate number of elements in the sample.

In statistical studies, biases must be avoided. *Bias* is an error that causes the study to favor one set of results over another. For example, if a survey to determine how the country views the president's job performance only speaks to registered voters in the president's party, the results will be skewed because a disproportionately large number of responders would tend to show approval, while a disproportionately large number of people in the opposite party would tend to express disapproval.

Extraneous Variables are, as the name implies, outside influences that can affect the outcome of a study. They are not always avoidable, but could trigger bias in the result.

Data analysis

The *Measure of Central Tendency* is a statistical value that gives a general tendency for the center of a group of data. There are several different ways of describing the measure of central tendency. Each one has a unique way it is calculated, and each one gives a slightly different perspective on the data set. Whenever you give a measure of central tendency, always make sure the units are the same. If the data has different units, such as hours, minutes, and seconds, convert all the data to the same unit, and use the same unit in the measure of central tendency. If no units are given in the data, do not give units for the measure of central tendency.

The statistical *Mean* of a group of data is the same as the arithmetic average of that group. To find the mean of a set of data, first convert each value to the same units, if necessary. Then find the sum of all the values, and count the total number of data values, making sure you take into consideration each individual value. If a value appears more than once, count it more than once. Divide the sum of the values by the total number of values and apply the units, if any. Note that the mean does not have to be one of the data values in the set, and may not divide evenly.

$$\text{mean} = \frac{\text{sum of the data values}}{\text{quantity of data values}}$$

While the mean is relatively easy to calculate and averages are understood by most people, the mean can be very misleading if used as the sole measure of central tendency. If the data set has outliers (data values that are unusually high or unusually low compared to the rest of the data values), the mean can be very distorted, especially if the data set has a small number of values. If unusually high values are countered with unusually low values, the mean is not affected as much. For example, if five of twenty students in a class get a 100 on a test, but the other 15 students have an average of 60 on the same test, the class average would appear as 70. Whenever the mean is skewed by outliers, it is always a good idea to include the median as an alternate measure of central tendency.

The statistical *Median* is the value in the middle of the set of data. To find the median, list all data values in order from smallest to largest or from largest to smallest. Any value that is repeated in the set must be listed the number of times it appears. If there are an odd number of data values, the median is the value in the middle of the list. If there is an even number of data values, the median is the arithmetic mean of the two middle values.

The statistical *Mode* is the data value that occurs the most number of times in the data set. It is possible to have exactly one mode, more than one mode, or no mode. To find the mode of a set of data, arrange the data like you do to find the median (all values in order, listing all multiples of data values). Count the number of times each value appears in the data set. If all values appear an equal number of times, there is no mode. If one value appears more than any other value, that value is the mode. If two or more values appear the same number of times, but there are other values that appear fewer times and no values that appear more times, all of those values are the modes.

The big disadvantage of using the median as a measure of central tendency is that is relies solely on a value's relative size as compared to the other values in the set. When the individual values in a set of data are evenly dispersed, the median can be an accurate tool. However, if there is a group of rather large values or a group of rather small values that are not offset by a different group of values, the information that can be inferred from the median may not be accurate because the distribution of values is skewed.

The main disadvantage of the mode is that the values of the other data in the set have no bearing on the mode. The mode may be the largest value, the smallest value, or a value anywhere in between in the set. The mode only tells which value or values, if any, occurred the most number of times. It does not give any suggestions about the remaining values in the set.

The *Measure of Dispersion* is a single value that helps to "interpret" the measure of central tendency by providing more information about how the data values in the set are distributed about the measure of central tendency. The measure of dispersion helps to eliminate or reduce the disadvantages of using the mean, median, or mode as a single measure of central tendency, and give a more accurate picture of the data set as a whole. To have a measure of dispersion, you must know or calculate the range, standard deviation, or variance of the data set.

The *Range* of a set of data is the difference between the greatest and lowest values of the data in the set. To calculate the range, you must first make sure the units for all data values are the same, and then identify the greatest and lowest values. Use the formula range = highest value – lowest value. If there are multiple data values that are equal for the highest or lowest, just use one of the values in the formula. Write the answer with the same units as the data values you used to do the calculations.

Standard Deviation is a measure of dispersion that compares all the data values in the set to the mean of the set to give a more accurate picture. To find the standard deviation of a population, use the formula

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

where σ is the standard deviation of a population, x represents the individual values in the data set, $\bar{x}$ is the mean of the data values in the set, and n is the number of data values in the set. The higher the value of the standard deviation is, the greater the variance of the data values from the mean. The units associated with the standard deviation are the same as the units of the data values.

The *Variance* of a population, or just variance, is the square of the standard deviation of that population. While the mean of a set of data gives the average of the set and gives information about where a specific data value lies in relation to the average, the variance of the population gives information about the degree to which the data values are spread out and tell you how close an individual value is to the average compared to the other values. The units associated with variance are the same as the units of the data values squared.

Percentiles and Quartiles are other methods of describing data within a set. *Percentiles* tell what percentage of the data in the set fall below a specific point. For example, achievement test scores are often given in percentiles. A score at the 80th percentile is one which is equal to or higher than 80 percent of the scores in the set. In other words, 80 percent of the scores were lower than that score.

Quartiles are percentile groups that make up quarter sections of the data set. The first quartile is the 25th percentile. The second quartile is the 50th percentile; this is also the median of the data set. The third quartile is the 75th percentile.

Skewness is a way to describe the symmetry or asymmetry of the distribution of values in a data set. If the distribution of values is symmetrical, there is no skew. In general the closer the mean of a data set is to the median of the data set, the less skew there is. Generally, if the mean is to the right of the median, the data set is *Positively Skewed*, or right-skewed, and if the mean is to the left of the median, the data set is *negatively skewed*, or left-skewed. However, this rule of thumb is not infallible. When the data values are graphed on a curve, a set with no skew will be a perfect bell curve. To estimate skew, use the formula

$$\text{skew} = \frac{\sqrt{n(n-1)}}{n-2}\left(\frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{3}{2}}}\right)$$

where n is the number of values is the set, x_i is the ith value in the set, and $\bar{x}$ is the mean of the set.

In statistics, *Simple Regression* is using an equation to represent a relation between an independent and dependent variables. The independent variable is also referred to as the explanatory variable or the predictor, and is generally represented by the variable x in the equation. The dependent variable, usually represented by the variable y, is also referred to as the response variable. The equation may be any type of function – linear, quadratic, exponential, etc. The best way to handle this task is to use the regression feature of your graphing calculator. This will easily give you the curve of best fit and provide you with the coefficients and other information you need to derive an equation.

In a scatter plot, the *Line of Best Fit* is the line that best shows the trends of the data. The line of best fit is given by the equation $\hat{y} = ax + b$, where a and b are the regression coefficients. The regression coefficient a is also the slope of the line of best fit, and b is also the y-coordinate of the point at which the line of best fit crosses the x-axis. Not every point on the scatter plot will be on the line of best fit. The differences between the y-values of the points in the scatter plot and the corresponding y-values according to the equation of the line of best fit are the residuals. The line of best fit is also called the least-squares regression line because it is also the line that has the lowest sum of the squares of the residuals.

The *Correlation Coefficient* is the numerical value that indicates how strong the relationship is between the two variables of a linear regression equation. A correlation coefficient of

−1 is a perfect negative correlation. A correlation coefficient of +1 is a perfect positive correlation. Correlation coefficients close to −1 or +1 are very strong correlations. A correlation coefficient equal to zero indicates there is no correlation between the two variables. This test is a good indicator of whether or not the equation for the line of best fit is accurate. The formula for the correlation coefficient is

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where r is the correlation coefficient, n is the number of data values in the set, (x_i, y_i) is a point in the set, and $\bar{x}$ and $\bar{y}$ are the means.

A *Z-score* is an indication of how many standard deviations a given value falls from the mean. To calculate a z-score, use the formula $= \frac{x-\mu}{\sigma}$, where x is the data value, μ is the mean of the data set, and σ is the standard deviation of the population. If the z-score is positive, the data value lies above the mean. If the z-score is negative, the data value falls below the mean. These scores are useful in interpreting data such as standardized test scores, where every piece of data in the set has been counted, rather than just a small random sample. In cases where standard deviations are calculated from a random sample of the set, the z-scores will not be as accurate.

According to the *Central Limit Theorem*, regardless of what the original distribution of a sample is, the distribution of the means tends to get closer and closer to a normal distribution as the sample size gets larger and larger (this is necessary because the sample is becoming more all-encompassing of the elements of the population). As the sample size gets larger, the distribution of the sample mean will approach a normal distribution with a mean of the population mean and a variance of the population variance divided by the sample size.

Displaying nformation

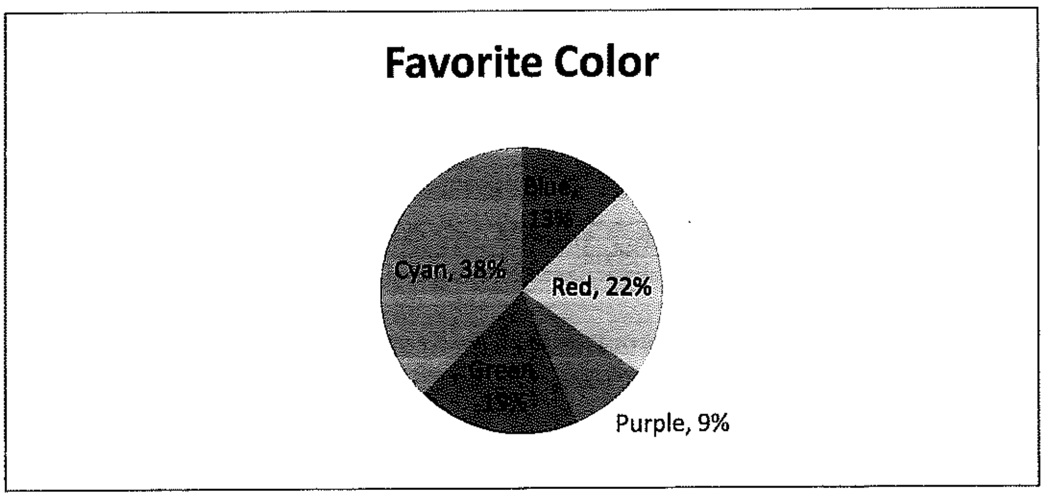
Charts and *Tables* are ways of organizing information into separate rows and columns that are labeled to identify and explain the data contained in them. Some charts and tables are organized horizontally, with row lengths giving the details about the labeled information. Other charts and tables are organized vertically, with column heights giving the details about the labeled information.

Frequency Tables show how frequently each unique value appears in the set. A *Relative Frequency Table* is one that shows the proportions of each unique value compared to the entire set. Relative frequencies are given as percents; however, the total percent for a relative frequency table will not necessarily equal 100 percent due to rounding. An example of a frequency table with relative frequencies is below.

Favorite Color	Frequency	Relative Frequency
Blue	4	13%
Red	7	22%
Purple	3	9%
Green	6	19%
Cyan	12	38%

A *Pictograph* is a graph, generally in the horizontal orientation, that uses pictures or symbols to represent the data. Each pictograph must have a key that defines the picture or symbol and gives the quantity each picture or symbol represents. Pictures or symbols on a pictograph are not always shown as whole elements. In this case, the fraction of the picture or symbol shown represents the same fraction of the quantity a whole picture or symbol stands for. For example, a row with $3\frac{1}{2}$ ears of corn, where each ear of corn represents 100 stalks of corn in a field, would equal $3\frac{1}{2} \cdot 100 = 350$ stalks of corn in the field.

Circle Graphs, also known as *Pie Charts*, provide a visual depiction of the relationship of each type of data compared to the whole set of data. The circle graph is divided into sections by drawing radii to create central angles whose percentage of the circle is equal to the individual data's percentage of the whole set. Each 1% of data is equal to 3.6° in the circle graph. Therefore, data represented by a 90° section of the circle graph makes up 25% of the whole. When complete, a circle graph often looks like a pie cut into uneven wedges. The pie chart below shows the data from the frequency table referenced earlier where people were asked their favorite color.



➢ **Review Video: <u>Pie Chart</u>**
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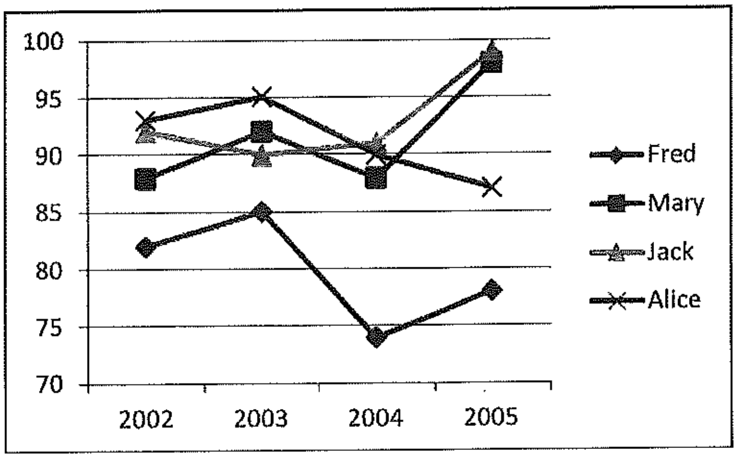
Line Graphs have one or more lines of varying styles (solid or broken) to show the different values for a set of data. The individual data are represented as ordered pairs, much like on a Cartesian plane. In this case, the *x*- and *y*- axes are defined in terms of their units, such as dollars or time. The individual plotted points are joined by line segments to show whether the value of the data is increasing (line sloping upward), decreasing (line sloping downward) or staying the same (horizontal line). Multiple sets of data can be graphed on the same line graph to give an easy visual comparison. An example of this would be graphing achievement test scores for different groups of students over the same time period to see which group had the greatest increase or decrease in performance from year-to-year (as shown below).

> ➤ **Review Video: Line Graphs**
> *Visit **mometrix.com/academy** and enter **Code**:* **480147**



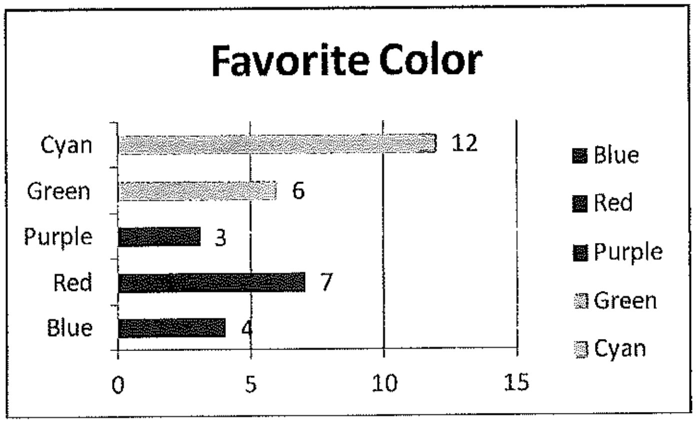
A *Line Plot*, also known as a *Dot Plot*, has plotted points that are NOT connected by line segments. In this graph, the horizontal axis lists the different possible values for the data, and the vertical axis lists the number of times the individual value occurs. A single dot is graphed for each value to show the number of times it occurs. This graph is more closely related to a bar graph than a line graph. Do not connect the dots in a line plot or it will misrepresent the data.

> ➤ **Review Video: Line Plot**
> *Visit **mometrix.com/academy** and enter **Code**:* **754610**

A *Stem and Leaf Plot* is useful for depicting groups of data that fall into a range of values. Each piece of data is separated into two parts: the first, or left, part is called the stem; the second, or right, part is called the leaf. Each stem is listed in a column from smallest to largest. Each leaf that has the common stem is listed in that stem's row from smallest to largest. For example, in a set of two-digit numbers, the digit in the tens place is the stem, and the digit in the ones place is the leaf. With a stem and leaf plot, you can easily see which subset of numbers (10s, 20s, 30s, etc.) is the largest. This information is also readily available by looking at a histogram, but a stem and leaf plot also allows you to look closer and see exactly which values fall in that range. Using all of the test scores from above, we can assemble a stem and leaf plot like the one below.

Test Scores									
7	4	8							
8	2	5	7	8	8				
9	0	0	1	2	2	3	5	8	9

A *Bar Graph* is one of the few graphs that can be drawn correctly in two different configurations – both horizontally and vertically. A bar graph is similar to a line plot in the way the data is organized on the graph. Both axes must have their categories defined for the graph to be useful. Rather than placing a single dot to mark the point of the data's value, a bar, or thick line, is drawn from zero to the exact value of the data, whether it is a number, percentage, or other numerical value. Longer bar lengths correspond to greater data values. To read a bar graph, read the labels for the axes to find the units being reported. Then look where the bars end in relation to the scale given on the corresponding axis and determine the associated value. The bar chart below represents the responses from our favorite color survey.



➢ **Review Video: <u>Bar Graph</u>**
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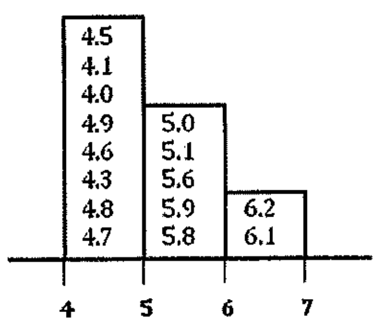
At first glance, a *Histogram* looks like a vertical bar graph. The difference is that a bar graph has a separate bar for each piece of data and a histogram has one continuous bar for each *Range* of data. For example, a histogram may have one bar for the range 0–9, one bar for 10–19, etc. While a bar graph has numerical values on one axis, a histogram has numerical values on both axes. Each range is of equal size, and they are ordered left to right from lowest to highest. The height of each column on a histogram represents the number of data values within that range. Like a stem and leaf plot, a histogram makes it easy to glance at the graph and quickly determine which range has the greatest quantity of values. A simple example of a histogram is below.



➢ **Review Video: <u>Histogram</u>**
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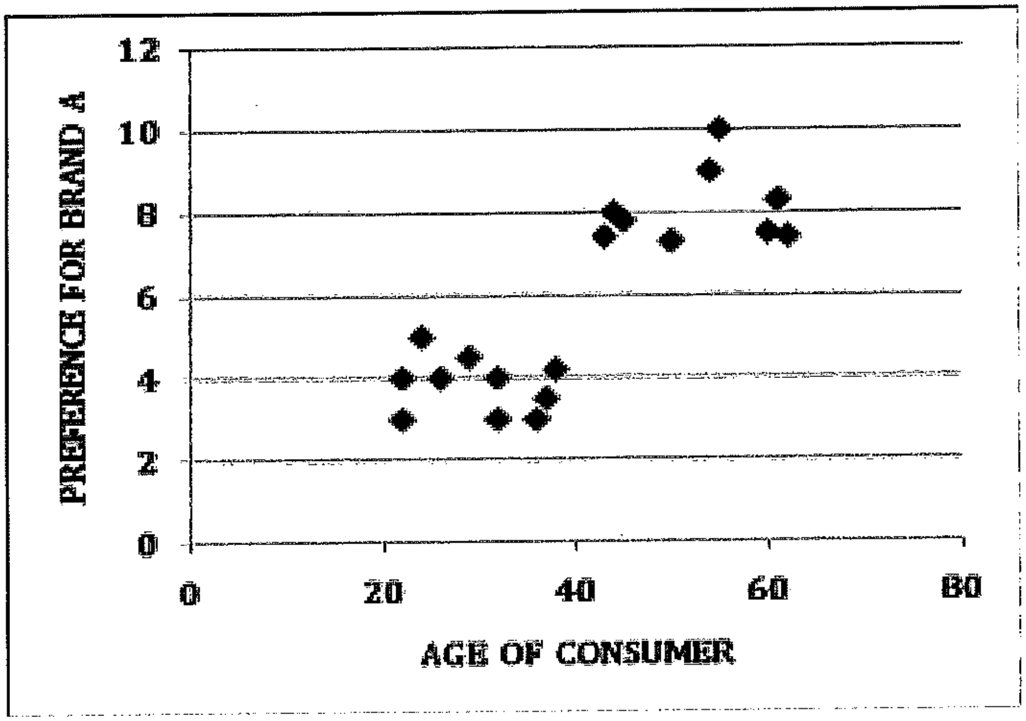
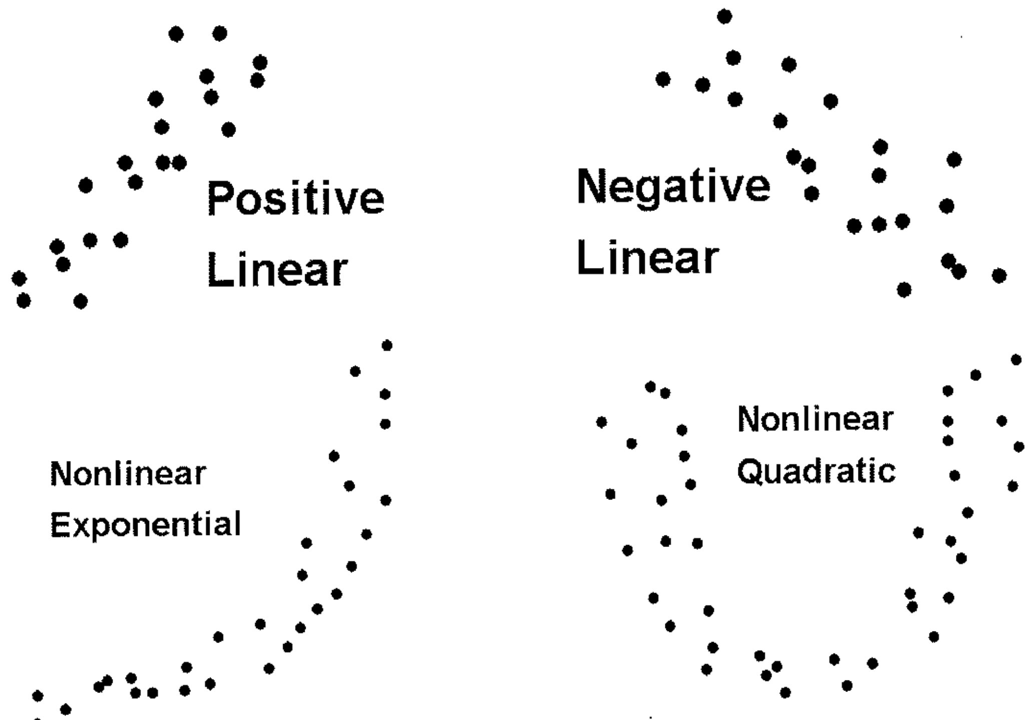
Bivariate Data is simply data from two different variables. (The prefix *bi-* means *two*.) In a *Scatter Plot*, each value in the set of data is plotted on a grid similar to a Cartesian plane, where each axis represents one of the two variables. By looking at the pattern formed by the points on the grid, you can often determine whether or not there is a relationship between the two variables, and what that relationship is, if it exists. The variables may be directly proportionate, inversely proportionate, or show no proportion at all. It may also be possible to determine if the data is linear, and if so, to find an equation to relate the two variables. The following scatter plot shows the relationship between preference for brand "A" and the age of the consumers surveyed.



Scatter Plots are also useful in determining the type of function represented by the data and finding the simple regression. Linear scatter plots may be positive or negative. Nonlinear scatter plots are generally exponential or quadratic. Below are some common types of scatter plots:



➢ **Review Video: <u>Scatter Plot</u>**
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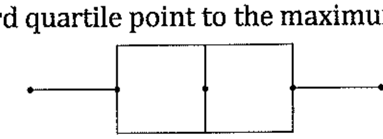
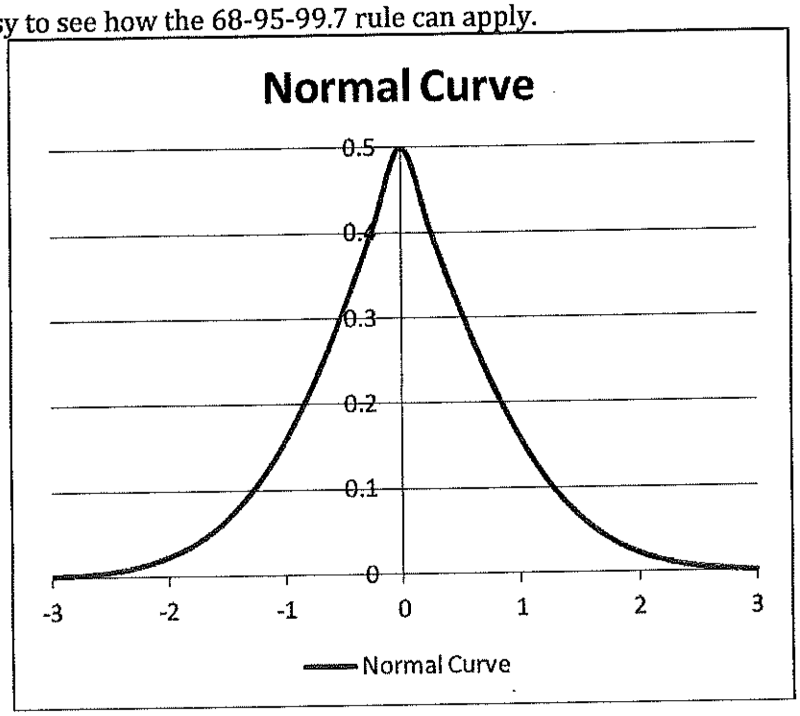
The *5-Number Summary* of a set of data gives a very informative picture of the set. The five numbers in the summary include the minimum value, maximum value, and the three quartiles. This information gives the reader the range and median of the set, as well as an indication of how the data is spread about the median.

A *Box-and-Whiskers Plot* is a graphical representation of the 5-number summary. To draw a box-and-whiskers plot, plot the points of the 5-number summary on a number line. Draw a box whose ends are through the points for the first and third quartiles. Draw a vertical line in the box through the median to divide the box in half. Draw a line segment from the first quartile point to the minimum value, and from the third quartile point to the maximum value.



The *68–95–99.7 Rule* describes how a normal distribution of data should appear when compared to the mean. This is also a description of a normal bell curve. According to this rule, 68 percent of the data values in a normally distributed set should fall within one standard deviation of the mean (34 percent above and 34 percent below the mean), 95 percent of the data values should fall within two standard deviations of the mean (47.5 percent above and 47.5 percent below the mean), and 99.7 percent of the data values should fall within three standard deviations of the mean, again, equally distributed on either side of the mean. This means that only 0.3 percent of all data values should fall more than three standard deviations from the mean. On the graph below, the normal curve is centered on the y-axis. The x-axis labels are how many standard deviations away from the center you are.

Therefore, it is easy to see how the 68-95-99.7 rule can apply.



The five general shapes of frequency curves are *Symmetrical, U-shaped, Skewed, J-shaped,* and *Multimodal.* Symmetrical curves are also known as bell curves or normal curves. Values equidistant from the median have equal frequencies. U-shaped curves have two maxima – one at each end. Skewed curves have the maximum point off-center. Curves that are negative skewed, or left skewed, have the maximum on the right side of the graph so there is longer tail and lower slope on the left side. The opposite is true for curves that are positive skewed, or right skewed. J-shaped curves have a maximum at one end and a minimum at the other end. Multimodal curves have multiple maxima. For example, if the curve has exactly two maxima, it is called a bimodal curve.

Factorials

The factorial is a function that can be performed on any non-negative integer. It is represented by the ! sign written after the integer on which it is being performed. The factorial of an integer is the product of all positive integers less than or equal to the number. For example, 4! (read "4 factorial") is calculated as $4 \times 3 \times 2 \times 1 = 24$.

Since 0 is not itself a positive integer, nor does it have any positive integers less than it, 0! cannot be calculated using this method. Instead, 0! is defined by convention to equal 1. This makes sense if you consider the pattern of descending factorials:

$$5! = 120$$
$$4! = \frac{5!}{5} = 24$$
$$3! = \frac{4!}{4} = 6$$
$$2! = \frac{3!}{3} = 2$$
$$1! = \frac{2!}{2} = 1$$
$$0! = \frac{1!}{1} = 1$$

Permutations and combinations

For any given set of data, the individual elements in the set may be arranged in different groups containing different numbers of elements arranged in different orders. For example, given the set of integers from one to three, inclusive, the elements of the set are 1, 2, and 3: written as {1, 2, 3}. They may be arranged as follows: 1, 2, 3, 12, 21, 13, 31, 23, 32, 123, 132, 213, 231, 312, and 321. These ordered sequences of elements from the given set of data are called permutations. It is important to note that in permutations, the order of the elements in the sequence is important. The sequence 123 is not the same as the sequence 213. Also, no element in the given set may be used more times as an element in a permutation than it appears as an element in the original set. For example, 223 is not a permutation in the above example because the number 2 only appears one time in the given set.

To find the number of permutations of *r* items from a set of *n* items, use the formula ${}_nP_r = \frac{n!}{(n-r)!}$. When using this formula, each element of *r* must be unique. Also, this assumes that different arrangements of the same set of elements yields different outcomes. For example, 123 is not the same as 321; order is important.

A special case arises while finding the number of possible permutations of *n* items from a set of *n* items. Because $n = r$, the equation for the number of permutations becomes simply $P = n!$.

If a set contains one or more groups of indistinguishable or interchangeable elements (e.g., the set {1, 2, 3, 3}, which has a group of two indistinguishable 3's), there is a different formula for finding distinct permutations of all *n* elements. Use the formula $P = \frac{n!}{m_1!m_2!...m_k!}$, where *P* is the number of permutations, *n* is the total number of elements in the set, and m_1 through m_k are the number of identical elements in each group (e.g., for the set {1, 1, 2, 2, 2, 3, 3}, $m_1 = 2$, $m_2 = 3$, and $m_3 = 2$). It is important to note that each repeated number is counted as its own element for the purpose of defining *n* (e.g., for the set {1, 1, 2, 2, 2, 3, 3}, $n = 7$, not 3).

To find the number of possible permutations of any number of elements in a set of unique elements, you must apply the permutation formulas multiple times. For example, to find the total number of possible permutations of the set {1, 2, 3} first apply the permutation formula for situations where $n = r$ as follows: $P = n! = 3! = 6$. This gives the number of permutations of the three elements when all three elements are used. To find the number of permutations when only two of the three elements are used, use the formula ${}_nP_r = \frac{n!}{(n-r)!}$, where n is 3 and r is 2.

$${}_nP_r = \frac{n!}{(n-r)!} \Rightarrow {}_3P_2 = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

To find the number of permutations when one element is used, use the formula ${}_nP_r = \frac{n!}{(n-r)!}$, where n is 3 and r is 1.

$${}_nP_r = \frac{n!}{(n-r)!} \Rightarrow {}_3P_1 = \frac{3!}{(3-1)!} = \frac{3!}{2!} = \frac{6}{2} = 3$$

Find the sum of the three formulas: $6 + 6 + 3 = 15$ total possible permutations.

Alternatively, the general formula for total possible permutations can be written as follows:

$$P_T = \sum_{i=1}^{n} \frac{n!}{(i-1)!}$$

Combinations are essentially defined as permutations where the order in which the elements appear does not matter. Going back to the earlier example of the set {1, 2, 3}, the possible combinations that can be made from that set are 1, 2, 3, 12, 13, 23, and 123.

In a set containing n elements, the number of combinations of r items from the set can be found using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$. Notice the similarity to the formula for permutations. In effect, you are dividing the number of permutations by $r!$ to get the number of combinations, and the formula may be written ${}_nC_r = \frac{{}_nP_r}{r!}$. When finding the number of combinations, it is important to remember that the elements in the set must be unique (i.e., there must not be any duplicate items), and that no item may be used more than once in any given sequence.

Probability

Probability is a branch of statistics that deals with the likelihood of something taking place. One classic example is a coin toss. There are only two possible results: heads or tails. The likelihood, or probability, that the coin will land as heads is 1 out of 2 (1/2, 0.5, 50%). Tails has the same probability. Another common example is a 6-sided die roll. There are six possible results from rolling a single die, each with an equal chance of happening, so the probability of any given number coming up is 1 out of 6.

Terms frequently used in probability:
Event – a situation that produces results of some sort (a coin toss)
Compound event – event that involves two or more independent events (rolling a pair of dice; taking the sum)
Outcome – a possible result in an experiment or event (heads, tails)
Desired outcome (or success) – an outcome that meets a particular set of criteria (a roll of 1 or 2 if we are looking for numbers less than 3)

Independent events – two or more events whose outcomes do not affect one another (two coins tossed at the same time)
Dependent events – two or more events whose outcomes affect one another (two cards drawn consecutively from the same deck)
Certain outcome – probability of outcome is 100% or 1
Impossible outcome – probability of outcome is 0% or 0
Mutually exclusive outcomes – two or more outcomes whose criteria cannot all be satisfied in a single event (a coin coming up heads and tails on the same toss)

Probability is the likelihood of a certain outcome occurring for a given event. The theoretical probability can usually be determined without actually performing the event. The likelihood of a outcome occurring, or the probability of an outcome occurring, is given by the formula

$$P(A) = \frac{\text{Number of acceptable outcomes}}{\text{Number of possible outcomes}}$$

where $P(A)$ is the probability of an outcome A occurring, and each outcome is just as likely to occur as any other outcome. If each outcome has the same probability of occurring as every other possible outcome, the outcomes are said to be equally likely to occur. The total number of acceptable outcomes must be less than or equal to the total number of possible outcomes. If the two are equal, then the outcome is certain to occur and the probability is 1. If the number of acceptable outcomes is zero, then the outcome is impossible and the probability is 0.

> **Review Video: <u>Theoretical and Experimental Probability</u>**
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Example:
There are 20 marbles in a bag and 5 are red. The theoretical probability of randomly selecting a red marble is 5 out of 20, (5/20 = 1/4, 0.25, or 25%).

When trying to calculate the probability of an event using the $\frac{desired\ outcomes}{total\ outcomes}$ formula, you may frequently find that there are too many outcomes to individually count them. Permutation and combination formulas offer a shortcut to counting outcomes. A permutation is an arrangement of a specific number of a set of objects in a specific order. The number of permutations of *r* items given a set of *n* items can be calculated as ${}_nP_r = \frac{n!}{(n-r)!}$. Combinations are similar to permutations, except there are no restrictions regarding the order of the elements. While ABC is considered a different permutation than BCA, ABC and BCA are considered the same combination. The number of combinations of *r* items given a set of *n* items can be calculated as ${}_nC_r = \frac{n!}{r!(n-r)!}$ or ${}_nC_r = \frac{{}_nP_r}{r!}$.
Example: Suppose you want to calculate how many different 5-card hands can be drawn from a deck of 52 cards. This is a combination since the order of the cards in a hand does not matter. There are 52 cards available, and 5 to be selected. Thus, the number of different hands is ${}_{52}C_5 = \frac{52!}{5!\times 47!} = 2{,}598{,}960$.

Sometimes it may be easier to calculate the possibility of something not happening, or the complement of an event. Represented by the symbol $\bar{A}$, the complement of A is the probability that event A does not happen. When you know the probability of event A occurring, you can use the formula $P(\bar{A}) = 1 - P(A)$, where $P(\bar{A})$ is the probability of event A not occurring, and $P(A)$ is the probability of event A occurring.

The addition rule for probability is used for finding the probability of a compound event. Use the formula $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, where $P(A \text{ and } B)$ is the probability of both events occurring to find the probability of a compound event. The probability of both events occurring at the same time must be subtracted to eliminate any overlap in the first two probabilities.

Conditional probability is the probability of an event occurring once another event has already occurred. Given event A and dependent event B, the probability of event B occurring when event A has already occurred is represented by the notation $P(A|B)$. To find the probability of event B occurring, take into account the fact that event A has already occurred and adjust the total number of possible outcomes. For example, suppose you have ten balls numbered 1–10 and you want ball number 7 to be pulled in two pulls. On the first pull, the probability of getting the 7 is $\frac{1}{10}$ because there is one ball with a 7 on it and 10 balls to choose from. Assuming the first pull did not yield a 7, the probability of pulling a 7 on the second pull is now $\frac{1}{9}$ because there are only 9 balls remaining for the second pull.

The multiplication rule can be used to find the probability of two independent events occurring using the formula $P(A \text{ and } B) = P(A) \times P(B)$, where $P(A \text{ and } B)$ is the probability of two independent events occurring, $P(A)$ is the probability of the first event occurring, and $P(B)$ is the probability of the second event occurring.
The multiplication rule can also be used to find the probability of two dependent events occurring using the formula $P(A \text{ and } B) = P(A) \times P(B|A)$, where $P(A \text{ and } B)$ is the probability of two dependent events occurring and $P(B|A)$ is the probability of the second event occurring after the first event has already occurred.
Before using the multiplication rule, you MUST first determine whether the two events are dependent or independent.

Use a combination of the multiplication rule and the rule of complements to find the probability that at least one outcome of the element will occur. This given by the general formula $P(\text{at least one event occurring}) = 1 - P(\text{no outcomes occurring})$. For example, to find the probability that at least one even number will show when a pair of dice is rolled, find the probability that two odd numbers will be rolled (no even numbers) and subtract from one. You can always use a tree diagram or make a chart to list the possible outcomes when the sample space is small, such as in the dice-rolling example, but in most cases it will be much faster to use the multiplication and complement formulas.

Expected value is a method of determining expected outcome in a random situation. It is really a sum of the weighted probabilities of the possible outcomes. Multiply the probability of an event occurring by the weight assigned to that probability (such as the amount of money won or lost). A practical application of the expected value is to determine whether a game of chance is really fair. If the sum of the weighted probabilities is equal to zero, the game is generally considered fair because the player has a fair chance to at least to break even. If the expected value is less than zero, then players lose more than they win. For example, a lottery drawing might allow the player to choose any three-digit number, 000–999. The probability of choosing the winning number is 1:1000. If it costs \$1 to play, and a winning number receives \$500, the expected value is $\left(-\$1 \cdot \frac{999}{1{,}000}\right) + \left(\$500 \cdot \frac{1}{1{,}000}\right) = -0.499 \text{ or } -\$0.50$. You can expect to lose on average 50 cents for every dollar you spend.

Most of the time, when we talk about probability, we mean theoretical probability. Empirical probability, or experimental probability or relative frequency, is the number of times an outcome occurs in a particular experiment or a certain number of observed events. While theoretical probability is based on what *should* happen, experimental probability is based on what *has* happened. Experimental probability is calculated in the same way as theoretical, except that actual outcomes are used instead of possible outcomes.

Theoretical and experimental probability do not always line up with one another. Theoretical probability says that out of 20 coin tosses, 10 should be heads. However, if we were actually to toss 20 coins, we might record just 5 heads. This doesn't mean that our theoretical probability is incorrect; it just means that this particular experiment had results that were different from what was predicted. A practical application of empirical probability is the insurance industry. There are no set functions that define life span, health, or safety. Insurance companies look at factors from hundreds of thousands of individuals to find patterns that they then use to set the formulas for insurance premiums.

Objective probability is based on mathematical formulas and documented evidence. Examples of objective probability include raffles or lottery drawings where there is a pre-determined number of possible outcomes and a predetermined number of outcomes that correspond to an event. Other cases of objective probability include probabilities of rolling dice, flipping coins, or drawing cards. Most gambling games are based on objective probability.

Subjective probability is based on personal or professional feelings and judgments. Often, there is a lot of guesswork following extensive research. Areas where subjective probability is applicable include sales trends and business expenses. Attractions set admission prices based on subjective probabilities of attendance based on varying admission rates in an effort to maximize their profit.

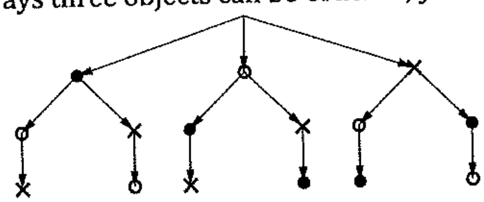
The total set of all possible results of a test or experiment is called a sample space, or sometimes a universal sample space. The sample space, represented by one of the variables S, Ω, or U (for universal sample space) has individual elements called outcomes. Other terms for outcome that may be used interchangeably include elementary outcome, simple event, or sample point. The number of outcomes in a given sample space could be infinite or finite, and some tests may yield multiple unique sample sets. For example, tests conducted by drawing playing cards from a standard deck would have one sample space of the card values, another sample space of the card suits, and a third sample space of suit-denomination combinations. For most tests, the sample spaces considered will be finite.

An event, represented by the variable E, is a portion of a sample space. It may be one outcome or a group of outcomes from the same sample space. If an event occurs, then the test or experiment will generate an outcome that satisfies the requirement of that event. For example, given a standard deck of 52 playing cards as the sample space, and defining the event as the collection of face cards, then the event will occur if the card drawn is a J, Q, or K. If any other card is drawn, the event is said to have not occurred.

For every sample space, each possible outcome has a specific likelihood, or probability, that it will occur. The probability measure, also called the distribution, is a function that assigns a real number probability, from zero to one, to each outcome. For a probability measure to be accurate, every outcome must have a real number probability measure that is greater than or equal to zero and less than or equal to one. Also, the probability measure of the sample space must equal one, and the probability measure of the union of multiple outcomes must equal the sum of the individual probability measures.

Probabilities of events are expressed as real numbers from zero to one. They give a numerical value to the chance that a particular event will occur. The probability of an event occurring is the sum of the probabilities of the individual elements of that event. For example, in a standard deck of 52 playing cards as the sample space and the collection of face cards as the event, the probability of drawing a specific face card is $\frac{1}{52} = 0.019$, but the probability of drawing any one of the twelve face cards is $12(0.019) = 0.228$. Note that rounding of numbers can generate different results. If you multiplied 12 by the fraction $\frac{1}{52}$ before converting to a decimal, you would get the answer $\frac{12}{52} = 0.231$.

For a simple sample space, possible outcomes may be determined by using a **tree diagram** or an organized chart. In either case, you can easily draw or list out the possible outcomes. For example, to determine all the possible ways three objects can be ordered, you can draw a tree diagram:



You can also make a chart to list all the possibilities:

First object	Second object	Third object
•	x	o
•	o	x
o	•	x
o	x	•
x	•	o
x	o	•

Either way, you can easily see there are six possible ways the three objects can be ordered.

If two events have no outcomes in common, they are said to be mutually exclusive. For example, in a standard deck of 52 playing cards, the event of all card suits is mutually exclusive to the event of all card values. If two events have no bearing on each other so that one event occurring has no influence on the probability of another event occurring, the two events are said to be independent. For example, rolling a standard six-sided die multiple times does not change that probability that a particular number will be rolled from one roll to the next. If the outcome of one event does affect the probability of the second event, the two events are said to be dependent. For example, if cards are drawn from a deck, the probability of drawing an ace after an ace has been drawn is different than the probability of drawing an ace if no ace (or no other card, for that matter) has been drawn.

In probability, the odds in favor of an event are the number of times the event will occur compared to the number of times the event will not occur. To calculate the odds in favor of an event, use the formula $\frac{P(A)}{1-P(A)}$, where $P(A)$ is the probability that the event will occur. Many times, odds in favor is given as a ratio in the form $\frac{a}{b}$ or $a{:}b$, where a is the probability of the event occurring and b is the complement of the event, the probability of the event not occurring. If the odds in favor are given as 2:5, that means that you can expect the event to occur two times for every 5 times that it does not occur. In other words, the probability that the event will occur is $\frac{2}{2+5} = \frac{2}{7}$.

In probability, the odds against an event are the number of times the event will not occur compared to the number of times the event will occur. To calculate the odds against an event, use the formula $\frac{1-P(A)}{P(A)}$, where $P(A)$ is the probability that the event will occur. Many times, odds against is given as a ratio in the form $\frac{b}{a}$ or $b{:}a$, where b is the probability the event will not occur (the complement of the event) and a is the probability the event will occur. If the odds against an event are given as 3:1, that means that you can expect the event to not occur 3 times for every one time it does occur. In other words, 3 out of every 4 trials will fail.

Instruction in Mathematics

Cognitive theorists and constructivists

Constructivists believe that students may construct knowledge by themselves. In other words, students are actively engaged in the construction of their own knowledge. Students will assimilate and accommodate in order to build new knowledge, based on previous knowledge. Thus, in planning instruction based on constructivism, a teacher would focus on grouping designs, environment, problem-solving tasks, and inclusion of multiple representations. The goal in such a classroom would be for students to construct knowledge on their own. There are different levels of constructivism, including weak constructivism and radical constructivism.

Cognitivists differ from constructivists in that they believe that active exploration is important in helping students make sense of observations and experiences. However, the students are not expected to invent or construct knowledge by themselves. They are only expected to make sense of the mathematics. In planning instruction based on cognitivism, a teacher would employ similar methods to those discussed above, with the focus on active exploration. Students would do a lot of comparisons of mathematical methods in making sense of ideas.

Constructivism

Three types of constructivism are weak constructivism, social constructivism, and radical constructivism. Weak constructivists believe that students construct their own knowledge, but also accept certain preconceived notions or facts. Social constructivists believe that students construct knowledge by interacting with one another and holding discussions and conversations. Radical constructivists believe that all interpretations of knowledge are subjective, based on the individual learner. In other words, there is no real truth; it is all subjective. Classroom instructional planning based on a weak constructivist viewpoint might involve incorporation of some accepted theorems and definitions, while continuing to plan active explorations and discussions. Planning based on a social constructivist viewpoint might involve group activities, debates, discussion forums, etc. Planning based on a radical constructivist viewpoint would involve activities that are open-ended, where there is more than one correct answer. The problems would invite more than one correct answer.

Project-based learning

Project-based learning is learning that centers on the solving of a problem. Students learn many different ideas by solving one "big" problem. For example, for a unit on sine and cosine functions, a teacher may design a problem whereby the students are asked to model a real-world phenomenon using both types of functions. Students must investigate the effects of changes in amplitude, period, shifts, etc., on the graphs of the functions. Students will also be able to make connections between the types of functions when modeling the same phenomenon. Such a problem will induce high-level thinking.

Project-based learning is derived from constructivist theory, which contends that students learn by doing and constructing their own knowledge.

Cooperative learning

Cooperative learning simply means that students will learn by cooperating with one another. Students will be placed into groups of a size determined by the teacher. With such an approach, students work together to succeed in learning. Students may work together to learn a topic, complete an assignment, or compete with other groups.

Examples of cooperative learning include Think-Pair-Share and Jigsaw. Think-Pair-Share is a cooperative learning strategy that involves thinking about some given topic, sharing ideas, thoughts, or questions with a partner, and then sharing the partner discussion with the whole group. For example, in the mathematics classroom, a teacher may ask the class to think about the meaning of a proportional relationship. Each student would think for a set period of time, share ideas with a partner, and then each partner group would share their ideas regarding the meaning of proportionality. Jigsaw is another cooperative learning strategy that involves dividing among each group member reading material or ideas to be learned. Each student will then read his or her information, summarize it, and share the findings or ideas with the group. In mathematics, students might be given information on modeling with cosine and sine functions. Students could then share what they learned about real-world phenomena modeled by each. Different students may also be assigned to read in-depth material on amplitude, period, shifts, etc.

Control strategies

"Control strategies" is another name for "metacognitive learning strategies," which indicate any strategy that promotes a learner's awareness of his or her level of learning. With such strategies, the student will work to determine what he or she knows and does not know regarding a subject. Possible control strategies are thinking, self-regulation, and discussing ideas with peers.

Example:

- A student may discover his or her level of "knowing" about functions by keeping a journal of any questions he or she might have regarding the topic. The student may list everything that he or she understands, as well as aspects not understood. As the student progresses through the course, he or she may go back and reconfirm any correct knowledge and monitor progress on any previous misconceptions.

Memorization and elaboration strategies

Memorization is simply a technique whereby rote repetition is used to learn information. Elaboration strategies involve the connection of new information to some previously learned information. In mathematics, for example, students may use elaboration strategies when learning how to calculate the volume of a cone, based on their understood approach for calculating the volume of a cylinder. The student would be making connections in his or her mind between this new skill and other previously acquired skills. A memorization technique would simply involve memorization of the volume of a cone formula, as well as ways to evaluate the formula.

Prior knowledge

Three ways of activating students' prior knowledge are concept mapping, visual imagery, and comparing and contrasting. With concept mapping, a student would detail and connect all known aspects of a mathematics topic. Ideas would be grouped into subgroups. Such an approach would allow a student to see what he or she does not know, prompting the activation of any prior knowledge on the subject. Visual imagery is simply the use of any pictures or diagrams to promote activation of prior knowledge. For example, giving a picture of Pascal's triangle would likely activate students' prior knowledge regarding the Binomial Theorem. Comparing and contrasting means that the student will compare and contrast ideas or approaches. For example, a student might be given a mapping of an inverse function. He or she could then compare and contrast this mapping to a known mapping of a function, in order to decide how they are the same and different. This would activate a student's prior understanding of functions and the definition thereof.

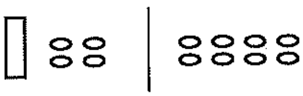
Three methods for ascertaining, or assessing, students' prior knowledge are portfolios, pre-tests, and self-inventories. Portfolios are simply a compilation of prior student activity related to mathematics topics. For example, a portfolio might show a student's work with transforming functions. Pre-tests are designed to measure a student's understanding of mathematics topics that will be taught in the course during the year. Self-inventories are just what the name implies: inventories that ask the students to name, list, describe, and explain information understood about various mathematics topics.

Once a teacher has assessed students' level of prior knowledge regarding some mathematics topic, he or she may use that information to scaffold the instruction. In other words, the teacher may decide to further break down the mathematics material into more integral parts. Exact processes or steps may be shown, including justification for using certain properties or theorems. More examples may be shown, while including examples of many different variations of problems, in order to ensure that students are not simply memorizing one approach that will be incorrectly applied to any problem of that sort. The teacher may also decide that more group work, peer cooperation, and discussion are needed.

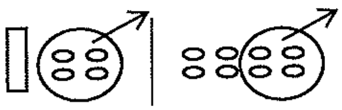
For example, suppose a teacher determines that students have very little understanding of logic and valid arguments. The teacher may decide to re-teach the creation of truth tables, including truth values for intersections and "if p, then q" statements. The teacher may also decide to re-teach how a truth table may be used to show if an argument is valid. Students may be placed into groups and asked to determine the validity of several simple arguments. Once students understand the concept, they may move on to more rigorous arguments, including equivalence relations.

Concept whereby usage of manipulatives would increase conceptual understanding

Understanding of how to solve one-variable equations would certainly be enhanced by using rods and counters. With this manipulative, the rod would represent the variable, or *x*, while the counters would represent the constants on each side of the equation. A sample diagram of the equation, $x + 4 = 8$, is shown below. Note that the vertical line represents the equals sign.



In order to solve the equation (and isolate x), four counters may be removed from each side of the mat. This process is shown below:



Now, the final illustration is:

Thus, the solution is $x = 4$. The manipulative helps students understand the meaning of the subtraction property of equality in action, without simply memorizing its meaning.

Problems

<u>Problem #1</u>

Explain how an understanding of the area under the normal curve may be supported by using a graphing calculator. Provide a sample problem and describe the steps involved in solving, using both a manual approach and a technological approach.

The area under the normal curve may be found by calculating z-scores for certain endpoint values. The mean to z areas for these z-scores may be used to find the area. A graphing calculator may use the normalpdf function and ShadeNorm function in order to show the same area under the normal curve, between two values. Consider the following problem: The class average on a statistics exam is 90, with a standard deviation of 4 points. Find the percentage of students who scored above 87 on the exam.

This problem may be solved manually by first calculating the z-score.

$$z = \frac{87 - 90}{4} = \frac{-3}{4}$$

Since the score falls below the mean, the area above the score will equal the sum of 0.5 (or the area of one-half of the normal curve) and the mean to z area, which is 0.2734. Thus, the area above the z-score of −0.75 is 0.7734. The percentage of students who scored above 87 was 77.34%.

This problem may also be solved by using the graphing calculator:

1. Enter normalpdf(x, 0, 1) into the y = screen. (This represents the normal curve having a mean of 0 and standard deviation of 1.)
2. Choose 2nd Vars, ShadeNorm(.
3. Enter ShadeNorm(87,100,90,4). (This represents the lower bound, upper bound, mean, and standard deviation.)
4. Record the area of approximately 0.77.
5. Thus, the calculator also shows that approximately 77% of the students scored above an 87.

Problem #2

Describe how the understanding of derivative and anti-derivative may be enhanced/supported using a graphing calculator. Provide an example and show how to solve, using a manual approach and technological approach.

The derivative of an expression is the slope of a tangent line to the curve, at a specific point. The anti-derivative of an expression is the inverse operation of the derivative. Taking the derivative of the anti-derivative will give the original expression. The derivative and anti-derivative can be calculated manually as shown below:

Given $f(x) = 3x^2 + 8x + 4$, the derivative is $f(x) = 6x + 8$. The anti-derivative is $f(x) = x^3 + 4x^2 + 4x$. Thus, evaluation of the derivative and anti-derivative for an x-value of 2 gives $f(x) = 20$ and $f(x) = 32$, respectively. The student can confirm his or her derivative and anti-derivative expressions by evaluating the graphed functions for the same x-value. If the expressions were correctly determined, then evaluation of the derivative and anti-derivative for the x-value should give the same y-value, for each.

Using the graphing calculator, the derivative and anti-derivative for a given point may be evaluated by entering the expression into the y = screen, graphing the function, selecting 2nd Trace, and then choosing dy/dx and $\int f(x)dx$. After selecting the derivative or anti-derivative, the x-value may be typed. Evaluation of the derivative or anti-derivative for that x-value will appear on the screen.

Piaget's cognitive development theory

Piaget's cognitive development theory is aligned with constructivism. In fact, constructivism is built on his ideas. Piaget's cognitive development theory indicates that students actively participate in the construction of their own knowledge via assimilation and accommodation. Current cognitive theorists do not believe that students have to construct their own knowledge, but instead that they only have to make sense of what they are observing.

The four stages of learning, as developed by Piaget, are sensorimotor, preoperational, concrete operational, and formal operational. The defined stages show the progression from concrete thinking to abstract thinking. In other words, a child would need an object to understand properties, in the first stage. By the fourth stage, the child would be able to think abstractly, without some concrete form. In mathematics, this idea might be illustrated by first working with diagrams and manipulatives of numbers and then later writing symbolic forms of the numbers, including the numerals. This would illustrate the progression from 0 to 7 years. In the years of 11 to adulthood, much deeper abstraction is utilized. For example, people would be able to discuss functions and general properties, without looking at any concrete graphs or representations.

Progression that a student undergoes as he or she learns mathematics

When learning mathematics, students begin with concrete representations and ideas. Later, students are able to abstract meaning and make generalizations. Students will also be asked to apply abstract ideas from one topic to another mathematics topic. In other words, students would move from concrete representations, ideas, and facts to symbolic representations and generalizations. Piaget outlined such a progression in his general four stages of cognitive learning. For example, a student may first learn about solving equations by using a balance scale. After the student understands the process, he or she can solve alone, using the symbolic equations. He or she would also be able to describe the process for solving any equation.

Direct instruction versus student-centered instruction

Direct instruction is instruction whereby the teacher delivers all content knowledge to be learned, and students, more or less, passively listen. The teacher employs a step-by-step instruction method for learning content. Student-centered instruction is learning whereby the teacher serves as a facilitator of learning and students actively participate in their own learning. Research has shown that students show a higher level of procedural and conceptual understanding when learning in a student-centered approach. Direct instruction might be more appropriate when teaching basic or fundamental theorems. Student-centered learning might be more appropriate when helping students make connections or develop higher-level thinking regarding a topic.

Cooperative learning task versus traditional task

Think-Pair-Share is an activity whereby a topic is first given for consideration on an individual basis. Next, the students are arranged in pairs and asked to discuss the topic (e.g., any questions, comments, generalizations, etc.). Finally, each pair will contribute to a whole-class discussion on the topic.

In mathematics, students would likely develop a higher level of understanding by using such an activity as Think-Pair-Share when learning about trigonometric functions. For example, students might be asked to consider different real-world situations that may be modeled with sine and cosine functions. Students could individually make a list and then share with a partner. Each partner group could then contribute to a whole class list. This list could be used as a reference sheet.

Implementing technology in classroom instruction

Technology may be implemented in the mathematics classroom in many ways. For example, Excel may be used to perform regressions, calculate lines of best fit, calculate correlation coefficients, plot residuals, show convergence or divergence of a sequence, etc. Calculators may be used to evaluate and graph functions, find area under the normal curve, calculate combinations and permutations, perform statistical tests, etc. Graphing software, such as GeoGebra, may be used to graph and explore many shapes and functions. Students may also use it to graph reflections, rotations, translations, and dilations.

Modifying instruction to accommodate English-language learners

In mathematics specifically, instruction may be modified to include illustrations of ideas, in addition to given words. Audio may also be included for problem tasks. English-language learners may also be grouped with other fluent English-speaking students in order to assist with learning of the mathematics topic. Students will be able to hear the conversation, in addition to seeing the topic in print. In addition, problems may be broken down into smaller pieces, which can help the student focus on one step at a time. Further, additional one-on-one time with the teacher may be needed, whereby the teacher reads aloud and illustrates examples to be learned.

Mathematics question that is closed-ended and then rewritten in an open-ended manner

Closed-Ended:

- Look at the graph of $y = x^2 + 2$. Decide if the graph represents a function.

Open-Ended:

- Provide an example of an equation that represents a function. Provide an example of an equation that does not represent a function. Explain how the graphs of the two equations compare to one another.

The first question will elicit a simple, straightforward response, or "Yes, it is a function."

The second question prompts the student to come up with two equations and then describe how the graphs of the two equations would compare. There is more than one possible answer, and the student has to make a comparison as well.

Good questioning response techniques

A few good questioning response techniques are:

1. Make sure the wait time is sufficient;
2. Do not include leading prompts within questions;
3. Ask more questions based on student answers;
4. Confirm or restate correct student comments.

The key to good questioning response techniques is to show the student that his or her comments are important and to connect those comments to other student comments. The student should feel that he or she has made a contribution to the community of learners. A teacher should always ask a meaningful, thought-provoking question and provide sufficient time for the student to provide a meaningful and well-thought-out response. Student answers should lead to more questions and ideas and not serve as an endpoint.

NCTM categories of questions that teachers should ask

The Professional Standards describe five categories of questions that teachers should ask. These categories are: 1) working together to make sense of problems; 2) individually making sense of problems; 3) reasoning about mathematics; 4) providing conjectures about mathematics; and 5) making connections within and outside of mathematics. Sample questions include "What does this mean?," "How can you prove that?," and "What does this relate to?". Categories 4 and 5 are high level and include questions that prompt students to invent ideas and make meaningful connections.

Accountants and mathematical modeling

Accountants use mathematical modeling in a variety of ways. For example, an accountant models the future value of a certificate of deposit (CD) using the compound interest formula. An accountant also may fit a regression line to a client's overall savings over *x* years. An accountant may model tax payments with residual plots. Accountants may use past income tax returns to predict future tax expenses. Accountants may compare rates of return when investing in different mutual funds, by fitting and comparing regression lines.

Scientists and functions

Scientists use functions to model real-world phenomena. For example, scientists use quadratic functions to model the height of an object tossed into the air or dropped from a certain height. Scientists use sine and cosine functions to model real-world occurrences such as the depth of water at various times of the day, the movement of a pendulum, etc. Scientists use exponential functions to analyze and predict the number of bacteria present after *x* amount of time. Scientists also use functions when analyzing the time it takes a rocket to reach a destination.

Making mathematics relevant to students' lives

Teachers can make mathematics relevant to students, using a variety of strategies. Teachers may include items relevant and pertinent to students within question stems, such as including "iPad," "apps," and video game names. Teachers should pose questions that are similar to what students may have asked themselves, such as, "If I invest this much money in an account and save for *x* years, after how many years will I have *y* dollars?" Teachers should include real-world problems to solve, and not simply include rote solving of equations. Students should know what sorts of scenarios may be modeled with rational expressions. Many researchers believe that curricula should be centered on the "real world," with all facets of mathematics learning spawning from that center. In other words, students often know how to convert a decimal to a percentage, but when reading *The Wall Street Journal*, they may not be able to interpret a percentage yield.

Assessment tool

A mathematics assessment tool is used to assess a student's prior knowledge, current knowledge, skill set, procedural knowledge, conceptual understanding, depth of understanding, and ability to make abstractions and generalizations. Perhaps the most important purpose of such a tool is to help the student develop and modify instruction. A teacher may determine that students are ready to surpass the current lesson or need it to be much more scaffolded. A teacher may also use the assessment to track students' progress. For example, a portfolio might show students' initial understanding of functions and end with their work with function modeling.

When a teacher needs to decide on an appropriate assessment tool, he or she needs to consider the purpose of the assessment. For example, if the purpose of an assessment is to direct the instruction, a pre-test may be a good assessment to use. If the purpose of the assessment is to determine the level of student understanding, then a whole-class discussion may be desired. If the purpose of an assessment is to assess student understanding of a unit of material, then an exam would be appropriate. If a teacher wishes to analyze student understanding and ability to abstract knowledge, then a performance assessment may be used. If a teacher wishes to check off skills mastered by students, then a checklist would be appropriate.

Valid test

A test is valid if it tests what it is supposed to test. In other words, a test is valid if it appropriately covers the material it is supposed to cover. For example, a topic not taught in class should not be included on a valid test. In order to construct a valid test, a teacher should make a list of all standards covered during that time period. The teacher should also closely mirror the design of problems examined in class, for homework, and in group discussions. Finally, the teacher should make sure that there is an even balance of questions to cover all of the material.

Valid exam

In order to select a valid exam, a teacher should make sure that the test aligns with the objectives and standards covered in the unit. The teacher should also make sure that the test problems are similar to those covered during class time. The teacher should make sure the percentages of questions devoted to each objective are balanced. In order for a test to be valid, it must be reliable, meaning that it produces similar results with different groups. A teacher may wish to check the validity and reliability results of an exam.

In general, an exam is considered invalid if it does not measure what it is supposed to measure. The exam may include questions from another unit. It may include questions with different wording techniques, making it much more difficult. The exam may include representations different from those covered in class. An invalid exam would not be reliable, meaning the results would not be consistent with different administrations of the exam. Biased questions and wording may also make an exam invalid.

Assessing students' understanding of what has been taught

In order to assess thought processes, open-ended questions are needed. The teacher may wish to have students write an essay, write entries in a mathematics journal, undergo a performance task, or participate in a debate or discussion. The teacher may also design a pre-test that includes all constructed response questions. In particular, a performance task requires students to justify solutions, which provide the teacher with insight into students' understanding and reasoning. In general, the assessment should include questions that ask students to make abstractions and justify their thinking.

Testing issue

<u>A student claims that an exam is more difficult and includes more content than what was presented in class. How might a teacher determine if the student's claim is true?</u>
The teacher would need to make a list of all objectives and standards covered during the time period. The teacher would also need to compile all problems and examples covered in class and as

homework. Finally, the teacher would need to do a careful analysis of the wording of the problems covered in class and as homework. If any of these items are not aligned to the exam, the teacher would need to go back and re-teach the material, using the created test as a guide for instruction.

Performance task

A performance task allows the teacher to assess process as well as product, meaning that a teacher can assess students' thought processes as well as their final answer. The level of student learning will be much clearer when reviewing a performance task. A performance task goes beyond a multiple-choice format, allowing for oral and tactile-kinesthetic performances. Furthermore, a performance task may combine several mathematics concepts into one assessment instrument. This type of assessment often includes real-world problems, which helps the student connect mathematics to the outside world.

Formative and summative assessments

Formative assessments are those given during the learning process. Formative assessments provide the teacher with information related to a student's progress at various stages throughout a time period. Formative assessments are used to modify instruction as needed. In other words, formative assessments inform instruction. Summative assessments are those given at the end of a learning period. Summative assessments serve to measure the cumulative knowledge gained. Examples of formative assessments include quizzes, checklists, observations, and discussion. Examples of summative assessments include exams, portfolios, performance tasks, and standardized tests.

Four formative assessments include quizzes, checklists, observations, and discussion. Quizzes are often short assessments that may include multiple-choice items, short response items, or essay items. Quizzes are often administered following presentation of a portion of a mathematics unit. Checklists include a list of skills or concepts that should be mastered or understood. A teacher will check off all items mastered by a student. Observations are informal means of assessing students' understanding of a topic. A teacher may observe students' questions, engagement, and performance on projects. Discussion is another informal formative assessment. Discussions, both in groups and whole-class formats, allow the teacher to analyze students' thinking.

Four summative assessments include exams, portfolios, performance tasks, and standardized tests. Exams may include closed-ended or open-ended questions. Exams may be administered after each unit, semester, or at the end of the year. Portfolios include tasks created by a student and may include writing pieces and other large projects. Although the portfolio contains formative work, the tool itself may be used as a summative assessment piece. Performance tasks are large-scale problems that include many different components that relate to some big idea. For example, a student may be asked to formulate a plan for modeling a real-world phenomenon with a sine function. The student may be asked to explain how the function would change, given changes to the amplitude, period, shifts, etc. The student may then explain how these components would need to change to fit a new function. Standardized tests are tests that compare a student's performance to that of other students. They are often given at the end of the school year.

Scoring rubric

A strong rubric will include unique performance criteria for each bullet. In other words, a portion of one criteria statement should not be included in a portion of another criteria statement. Each criteria statement should be clearly delineated, describing exactly what the student must be able to do. Furthermore, a strong rubric will often have scoring options, ranging from 0 to 4. When designing the rubric, it is helpful to create a model student response that will warrant each rubric score. It is also helpful to provide a space to provide feedback to students.

Enhancing student understanding

In order for an assessment to enhance student understanding, it should provide an opportunity for the student to learn something. The assessment should be a learning opportunity for the student. It should prompt the student to think deeper about a mathematics topic. In other words, the student should think, "Okay. I understand this. I wonder how the process/solution would change if I did this." The assessment might prompt the student to ask deeper questions in the next class session or complete research on a certain topic. In order to create such an assessment, open-ended and challenging questions should be included on the exam. The exam should not consist of simple, lower-level, one-answer questions.

Testing mathematical misconceptions

In order to design such an assessment, the teacher should include mathematical error-type problems, whereby the student must look at a solution process or conjecture and determine if he or she agrees, of if and where an error occurred. The student would need to identify the error, correct it, and explain why it was erroneous. The assessment should include a variety of mathematical misconceptions. One solution process may include more than one error. A teacher may also simply ask students to participate in a collaborative learning activity, whereby the students must share ideas and thoughts regarding a new mathematical topic.

Assessing prior knowledge

Such a pre-test must not include any leading prompts. It should include open-ended and constructed-response items as well. A pre-test with solely multiple-choice items will not be sufficient, since a student has the option of guessing. The test should include higher-level questions that require connections within the field of mathematics. In other words, the questions should not all be mutually exclusive. They should build on one another. Finally, the test might include student error problems as well.

Assessing both procedural knowledge and conceptual understanding

The assessment should include rote, algorithmic-type problems, as well as those that ask the student to utilize higher-level thinking, abstractions, and generalizations. The test should include open-ended, constructed-response-type problems. A performance task is an excellent assessment for assessing a student's ability to solve a problem, while also examining the student's thought processes, rationales, etc. In order to assess both types of understanding, the assessment will need to ask students to justify and explain solutions. In other words, the assessments should include questions at both ends of Bloom's Taxonomy.

Pre-test and post-test

A post-test should be exactly the same as an administered pre-test. If the teacher is to compare the results of a post-test to a pre-test, then the test and testing conditions should be identical. The pre-test assesses students' prior knowledge, while a post-test assesses students post knowledge.

Comparing the results, side by side, allows the teacher to track student progress. The teacher may wish to add additional questions to the post-test, but the original questions should remain.

Assessment that will show what students do and do not know

The teacher should include questions that are straightforward, involve errors, require justification, and require shown work. A student self-assessment is one such tool that would show misconceptions, understood material, and advanced knowledge. The assessment should include more than multiple-choice questions. Designing a performance assessment with scaffolded questions, whereby only one solution may be found based on a previous answer, will also show students' exact level of understanding. A debate format is one type of assessment whereby the teacher will be able to see a student's level of understanding, as he or she seeks to respond with a rebuttal.

Assessment to hone in on any error patterns evident in students' work

A portfolio would be an excellent assessment for monitoring any student error patterns. The teacher would be able to track student errors as the course progressed. The teacher would be given insight into how, and if, errors improved, or if some knowledge was acquired but other knowledge was still incorrect. The portfolio might include a series of similar questions related to a certain topic. For example, a portfolio may include function transformation questions. A student's ability to transform functions may be tracked, starting with simple linear functions and ending with complex sine functions.

Components that must be present in an assessment that supports student learning

The assessment must require students to think deeper than what they have covered in class. It should prompt them to make connections between topics. It should invite different ways of thinking about problem solving. In other words, the student may think, "Okay. I have seen a similar version in class. This problem is slightly different, in that the parabola is shifted left. This is the opposite of shifting right, so I will add the constant to the *x*-term." The assessment will thus solidify the student's understanding of how to shift any function.

Using assessment results of assessments given to ELL learners in order to modify instruction

The teacher would be able to see if language itself is a barrier in learning. In other words, if the group of ELL students, as a whole, show difficulty with a mathematics topic, the teacher may deduce that the content was not clear due to minimal supporting pictures, diagrams, and auditory support. The teacher may decide to reteach the lesson, using more visual cues, verbal pronunciations, explicit vocabulary usage, and peer-group placement. Collaborative learning may be employed.

Questions that a teacher may ask after reviewing the results of an administered exam

The teacher may ask the following:

1. Did I cover the content in an explicit manner?
2. Did I show plenty of examples?
3. Did I use multiple representations when teaching the concepts?
4. Did I design instruction such as to accommodate all modes of learning?
5. Was the test valid?

6. Did students have an adequate amount of time to complete the test?
7. Why did some groups of students score lower or higher?
8. Did any biased questions affect the results?

How focus on career and college readiness affects assessment and instructional design

The focus on college and career-readiness standards prompts publishers and teachers to utilize more real-world problems in instruction and assessments. The focus in mathematics classrooms is shifting to more real-world, cumulative problems that require understanding of many different mathematics concepts in order to solve. Problems are related to science, finance, medicine, etc. The focus includes the ability to apply the algorithms to many different career situations. In summary, the recent focus shifts the instructional design to an application-based status.

Role of assessment in a classroom focused on cognitive instruction

A cognitively guided classroom would be similar to a constructivist classroom, in that active participation would be present. However, in a cognitive classroom (as advocated by current cognitive theorists), students are not required to invent their own knowledge. Instead, they must simply make sense of what they are observing and experiencing. They may be assisted by the teacher. Thus, the role of an assessment in such a classroom is to ascertain student thought processes. Such an assessment would ask students to describe thinking and perhaps make connections to other mathematics topics. The assessment must ascertain students' reasoning abilities.

Instructional cycle described by a learning theorist

The 5E Learning Model is based on the thinking of Jean Piaget. It is a constructivist learning model. Piaget believed that students construct their own knowledge via active participation and experiences. Problem solving is integral to student learning. The cycle is listed as engagement, exploration, explanation, elaboration, and evaluation. Thus, with active engagement and exploration, the student is able to develop his or her own explanation, use assimilation and accommodation to make sense of the information, and then evaluate the material and make conjectures, etc.