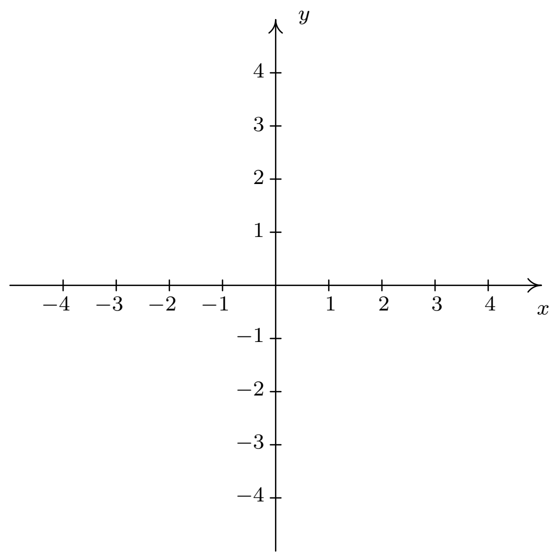
1.1.2 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[5] Imagine two real number lines crossing at a right angle at 0 as drawn below.



The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**.[6] As with the usual number line, we imagine these axes extending off indefinitely in both directions.[7] Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. This process is sometimes called 'projecting' the point P to the x- (respectively y-) axis. We then describe the point P using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**.[8] Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates**[9] of the point P. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to

[5]So named in honor of René Descartes.

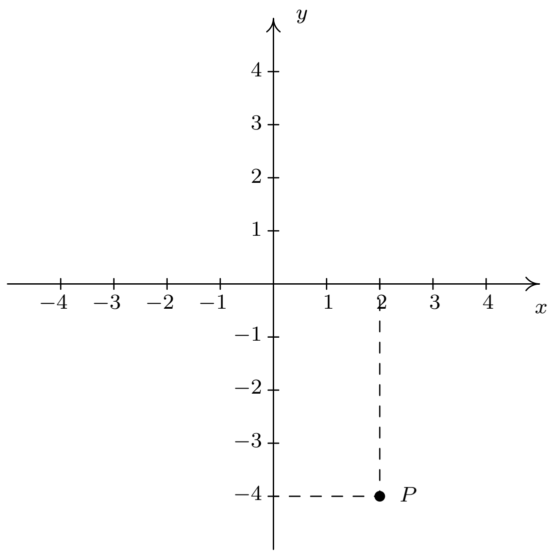
[6]The labels can vary depending on the context of application.

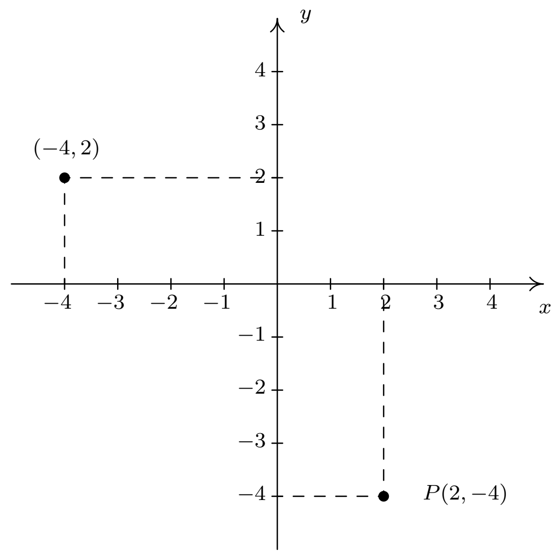
[7]Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y.

[8]Again, the names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the t-axis. The first number in the ordered pair would then be the t-coordinate.

[9]Also called the 'rectangular coordinates' of P – see Section 11.4 for more details.

reach P from the **origin** $(0,0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important − if we wish to plot the point $(-4,2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

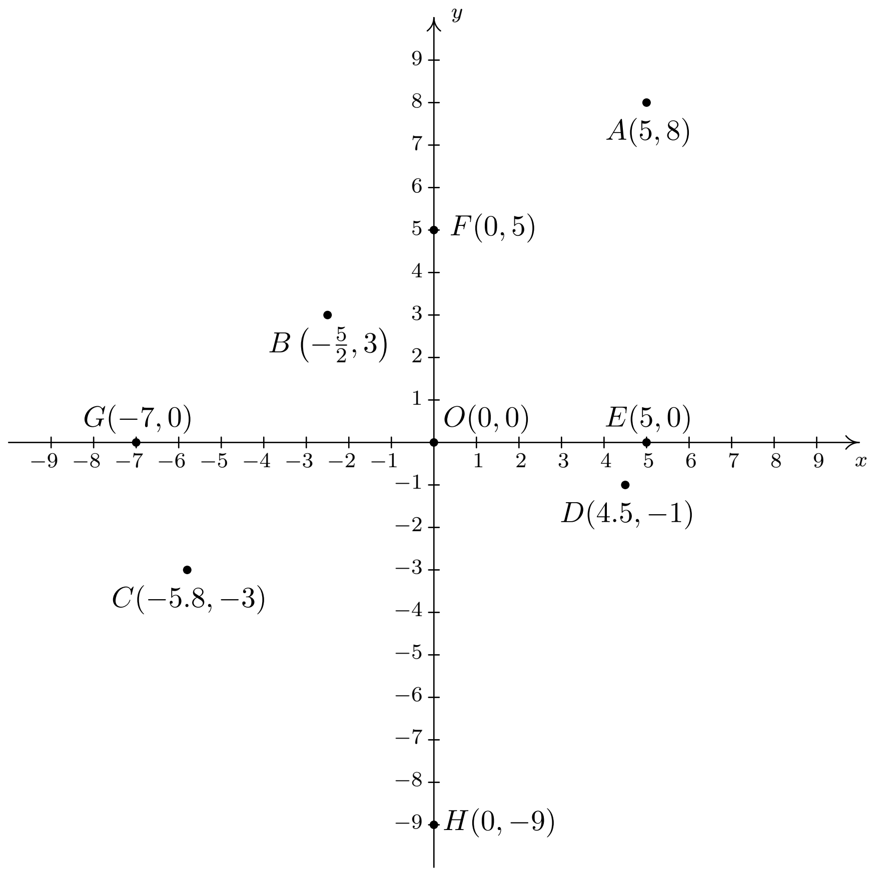
Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

Example 1.1.2. Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.[10]

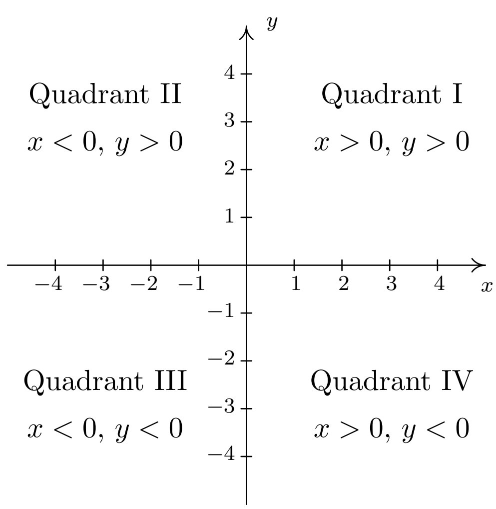
Solution. To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.

[10] The letter O is almost always reserved for the origin.



□

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



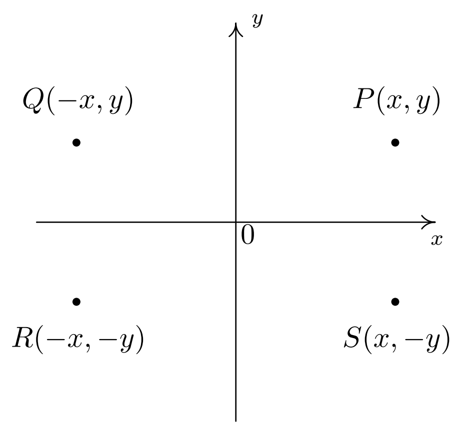
For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive y-axis whereas $(-117, 0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**.[11] There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.3. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the x-axis** if $a = c$ and $b = -d$
- **symmetric about the y-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

Example 1.1.3. Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

Check your answer by plotting the points.

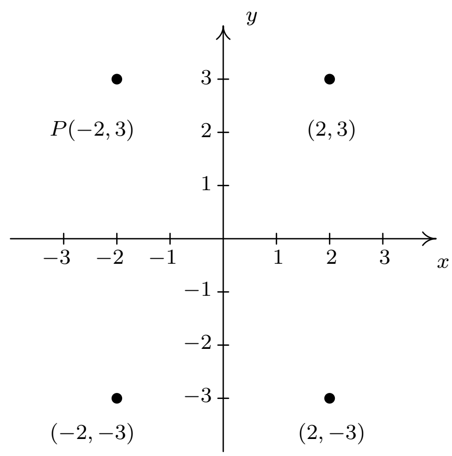
Solution. The figure after Definition 1.3 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

[11] According to Carl. Jeff thinks symmetry is overrated.

1. To find the point symmetric about the x-axis, we replace the y-coordinate with its opposite to get $(-2,-3)$.

2. To find the point symmetric about the y-axis, we replace the x-coordinate with its opposite to get $(2,3)$.

3. To find the point symmetric about the origin, we replace the x- and y-coordinates with their opposites to get $(2,-3)$.



□

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2,3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2,3)$ across the x-axis would lie at $(-2,-3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2,3)$ across that axis would be $(2,3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2,3)$ to $(-2,-3)$ then to $(2,-3)$, and so we would end up at the point symmetric to $(-2,3)$ about the origin. We summarize and generalize this process below.

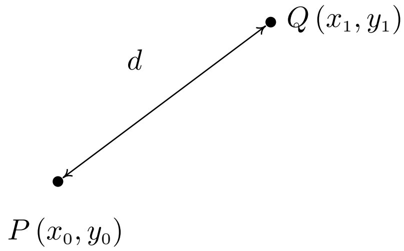
Reflections

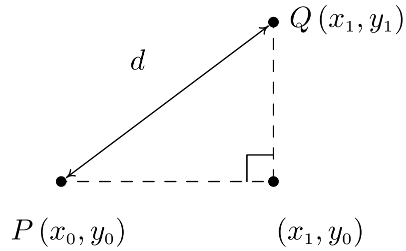
To reflect a point (x,y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.1.3 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P\left(x_0,y_0\right)$ and $Q\left(x_1,y_1\right)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.





With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$
$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation 1.1. The Distance Formula: The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 35 to verify Equation 1.1 for these cases.

Example 1.1.4. Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

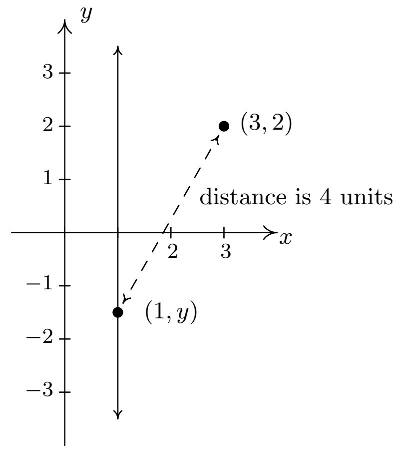
Solution.

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1-(-2))^2 + (-3-3)^2} \\ &= \sqrt{9+36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$. □

Example 1.1.5. Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution. We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$. Visually,

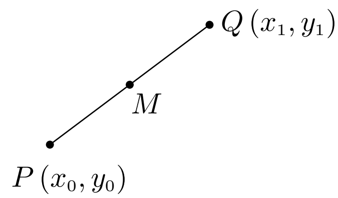


We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 1.1, yields

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}$$

We obtain two answers: $(1, 2 + 2\sqrt{3})$ and $(1, 2 - 2\sqrt{3})$. The reader is encouraged to think about why there are two answers. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.



If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Equation 1.2. The Midpoint Formula: The midpoint M of the line segment connecting $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right)$$

If we let d denote the distance between P and Q, we leave it as Exercise 36 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.6. Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution.

$$\begin{aligned} M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \\ &= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2} \right) = \left(-\frac{1}{2}, \frac{0}{2} \right) \\ &= \left(-\frac{1}{2}, 0 \right) \end{aligned}$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$. □

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 2.1.

Example 1.1.7. If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

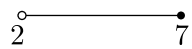
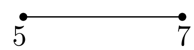
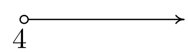
Solution. To prove the claim, we use Equation 1.2 to find the midpoint

$$\begin{aligned} M &= \left(\frac{a + b}{2}, \frac{b + a}{2} \right) \\ &= \left(\frac{a + b}{2}, \frac{a + b}{2} \right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required. □

1.1.4 EXERCISES

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 - 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

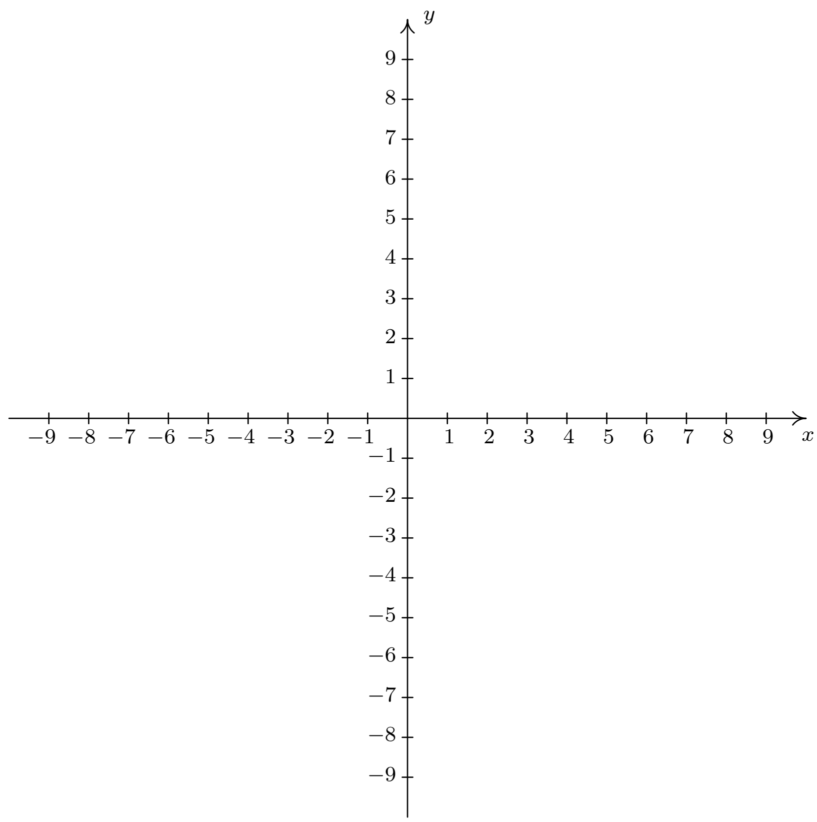
2. $(-1, 5] \cap [0, 8)$
3. $(-1, 1) \cup [0, 6]$
4. $(-\infty, 4] \cap (0, \infty)$
5. $(-\infty, 0) \cap [1, 5]$
6. $(-\infty, 0) \cup [1, 5]$
7. $(-\infty, 5] \cap [5, 8)$

In Exercises 8 - 19, write the set using interval notation.

8. $\{x \mid x \neq 5\}$
9. $\{x \mid x \neq -1\}$
10. $\{x \mid x \neq -3, 4\}$

11. $\{x \,|\, x \neq 0,\, 2\}$
12. $\{x \,|\, x \neq 2,\, -2\}$
13. $\{x \,|\, x \neq 0,\, \pm 4\}$
14. $\{x \,|\, x \leq -1 \text{ or } x \geq 1\}$
15. $\{x \,|\, x < 3 \text{ or } x \geq 2\}$
16. $\{x \,|\, x \leq -3 \text{ or } x > 0\}$
17. $\{x \,|\, x \leq 5 \text{ or } x = 6\}$
18. $\{x \,|\, x > 2 \text{ or } x = \pm 1\}$
19. $\{x \,|\, -3 < x < 3 \text{ or } x = 4\}$

20. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



21. For each point given in Exercise 20 above
    - Identify the quadrant or axis in/on which the point lies.
    - Find the point symmetric to the given point about the x-axis.
    - Find the point symmetric to the given point about the y-axis.
    - Find the point symmetric to the given point about the origin.

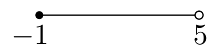
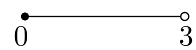
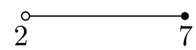
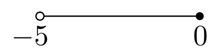
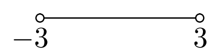
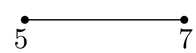
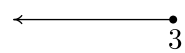
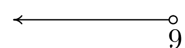
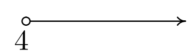
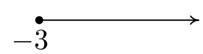
In Exercises 22 - 29, find the distance d between the points and the midpoint M of the line segment which connects them.

22. $(1, 2)$, $(-3, 5)$
23. $(3, -10)$, $(-1, 2)$
24. $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$
25. $\left(-\frac{2}{3}, \frac{3}{2}\right)$, $\left(\frac{7}{3}, 2\right)$
26. $\left(\frac{24}{5}, \frac{6}{5}\right)$, $\left(-\frac{11}{5}, -\frac{19}{5}\right)$.
27. $\left(\sqrt{2}, \sqrt{3}\right)$, $\left(-\sqrt{8}, -\sqrt{12}\right)$
28. $\left(2\sqrt{45}, \sqrt{12}\right)$, $\left(\sqrt{20}, \sqrt{27}\right)$.
29. $(0, 0)$, (x, y)
30. Find all of the points of the form $(x, -1)$ which are 4 units from the point $(3, 2)$.
31. Find all of the points on the y-axis which are 5 units from the point $(-5, 3)$.
32. Find all of the points on the x-axis which are 2 units from the point $(-1, 1)$.
33. Find all of the points of the form $(x, -x)$ which are 1 unit from the origin.
34. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
35. Verify the Distance Formula 1.1 for the cases when:
    (a) The points are arranged vertically. (Hint: Use $P(a, y_0)$ and $Q(a, y_1)$.)
    (b) The points are arranged horizontally. (Hint: Use $P(x_0, b)$ and $Q(x_1, b)$.)
    (c) The points are actually the same point. (You shouldn't need a hint for this one.)
36. Verify the Midpoint Formula by showing the distance between $P(x_1, y_1)$ and M and the distance between M and $Q(x_2, y_2)$ are both half of the distance between P and Q.
37. Show that the points A, B and C below are the vertices of a right triangle.
    (a) $A(-3, 2)$, $B(-6, 4)$, and $C(1, 8)$
    (b) $A(-3, 1)$, $B(4, 0)$ and $C(0, -3)$
38. Find a point $D(x, y)$ such that the points $A(-3, 1)$, $B(4, 0)$, $C(0, -3)$ and D are the corners of a square. Justify your answer.
39. Discuss with your classmates how many numbers are in the interval $(0, 1)$.
40. The world is not flat.[12] Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

[12]There are those who disagree with this statement. Look them up on the Internet some time when you're bored.

1.1.5 Answers

1.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

2. $(-1, 5] \cap [0, 8) = [0, 5]$

3. $(-1, 1) \cup [0, 6] = (-1, 6]$

4. $(-\infty, 4] \cap (0, \infty) = (0, 4]$

5. $(-\infty, 0) \cap [1, 5] = \emptyset$

6. $(-\infty, 0) \cup [1, 5] = (-\infty, 0) \cup [1, 5]$

7. $(-\infty, 5] \cap [5, 8) = \{5\}$

8. $(-\infty, 5) \cup (5, \infty)$

9. $(-\infty, -1) \cup (-1, \infty)$

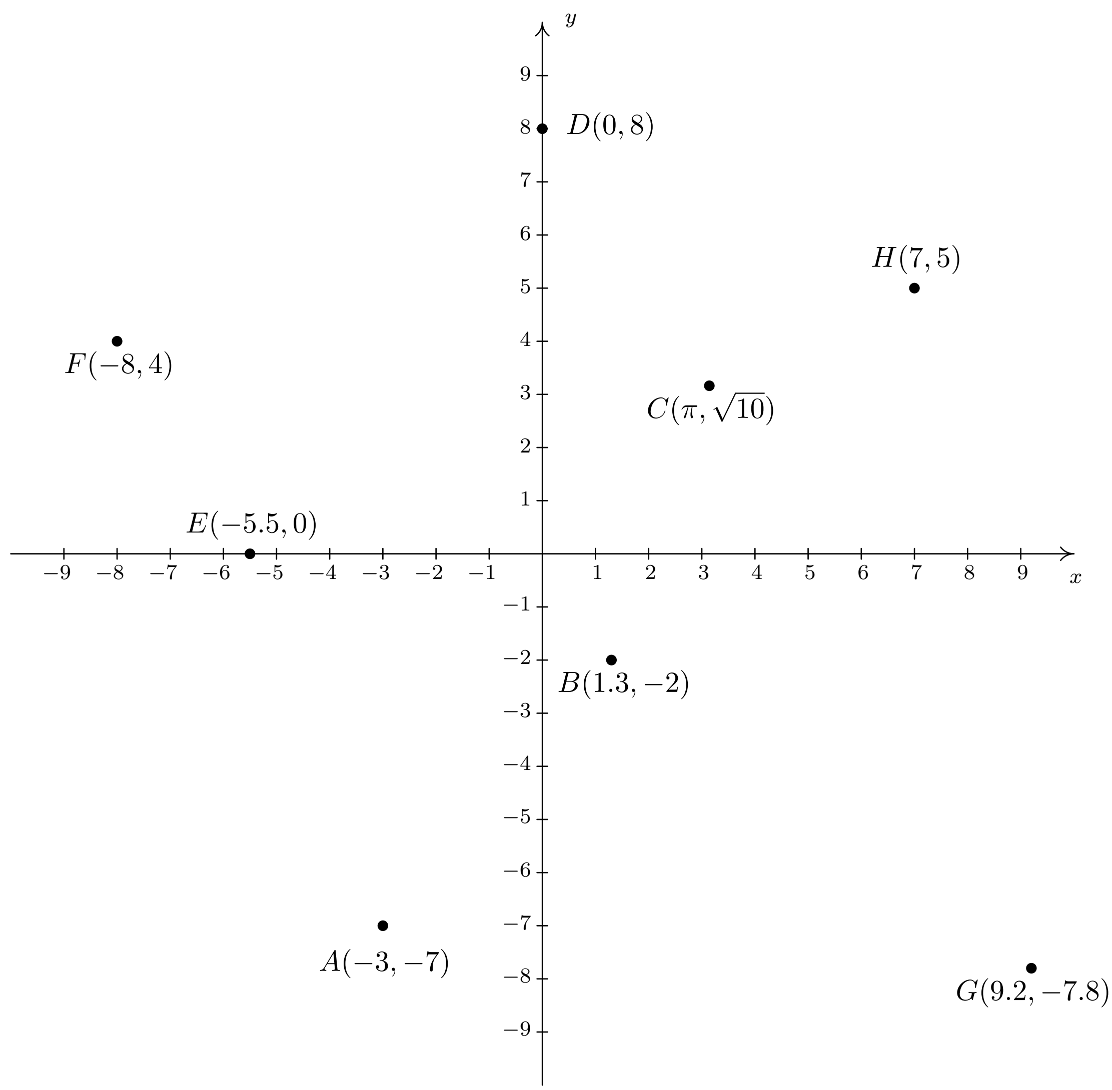
10. $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

11. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

12. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

13. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$

14. $(-\infty, -1] \cup [1, \infty)$
15. $(-\infty, \infty)$
16. $(-\infty, -3] \cup (0, \infty)$
17. $(-\infty, 5] \cup \{6\}$
18. $\{-1\} \cup \{1\} \cup (2, \infty)$
19. $(-3, 3) \cup \{4\}$
20. The required points $A(-3, -7)$, $B(1.3, -2)$, $C(\pi, \sqrt{10})$, $D(0, 8)$, $E(-5.5, 0)$, $F(-8, 4)$, $G(9.2, -7.8)$, and $H(7, 5)$ are plotted in the Cartesian Coordinate Plane below.



21. (a) The point $A(-3,-7)$ is
    - in Quadrant III
    - symmetric about x-axis with $(-3,7)$
    - symmetric about y-axis with $(3,-7)$
    - symmetric about origin with $(3,7)$

    (b) The point $B(1.3,-2)$ is
    - in Quadrant IV
    - symmetric about x-axis with $(1.3,2)$
    - symmetric about y-axis with $(-1.3,-2)$
    - symmetric about origin with $(-1.3,2)$

    (c) The point $C(\pi,\sqrt{10})$ is
    - in Quadrant I
    - symmetric about x-axis with $(\pi,-\sqrt{10})$
    - symmetric about y-axis with $(-\pi,\sqrt{10})$
    - symmetric about origin with $(-\pi,-\sqrt{10})$

    (d) The point $D(0,8)$ is
    - on the positive y-axis
    - symmetric about x-axis with $(0,-8)$
    - symmetric about y-axis with $(0,8)$
    - symmetric about origin with $(0,-8)$

    (e) The point $E(-5.5,0)$ is
    - on the negative x-axis
    - symmetric about x-axis with $(-5.5,0)$
    - symmetric about y-axis with $(5.5,0)$
    - symmetric about origin with $(5.5,0)$

    (f) The point $F(-8,4)$ is
    - in Quadrant II
    - symmetric about x-axis with $(-8,-4)$
    - symmetric about y-axis with $(8,4)$
    - symmetric about origin with $(8,-4)$

    (g) The point $G(9.2,-7.8)$ is
    - in Quadrant IV
    - symmetric about x-axis with $(9.2,7.8)$
    - symmetric about y-axis with $(-9.2,-7.8)$
    - symmetric about origin with $(-9.2,7.8)$

    (h) The point $H(7,5)$ is
    - in Quadrant I
    - symmetric about x-axis with $(7,-5)$
    - symmetric about y-axis with $(-7,5)$
    - symmetric about origin with $(-7,-5)$

22. $d=5$, $M=\left(-1,\frac{7}{2}\right)$

23. $d=4\sqrt{10}$, $M=(1,-4)$

24. $d=\sqrt{26}$, $M=\left(1,\frac{3}{2}\right)$

25. $d=\frac{\sqrt{37}}{2}$, $M=\left(\frac{5}{6},\frac{7}{4}\right)$

26. $d=\sqrt{74}$, $M=\left(\frac{13}{10},-\frac{13}{10}\right)$

27. $d=3\sqrt{5}$, $M=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{3}}{2}\right)$

28. $d=\sqrt{83}$, $M=\left(4\sqrt{5},\frac{5\sqrt{3}}{2}\right)$

29. $d=\sqrt{x^2+y^2}$, $M=\left(\frac{x}{2},\frac{y}{2}\right)$

30. $(3+\sqrt{7},-1)$, $(3-\sqrt{7},-1)$

31. $(0,3)$

32. $(-1+\sqrt{3},0)$, $(-1-\sqrt{3},0)$

33. $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

34. $(-3,-4)$, 5 miles, $(4,-4)$

37. (a) The distance from A to B is $|AB|=\sqrt{13}$, the distance from A to C is $|AC|=\sqrt{52}$, and the distance from B to C is $|BC|=\sqrt{65}$. Since $\left(\sqrt{13}\right)^2+\left(\sqrt{52}\right)^2=\left(\sqrt{65}\right)^2$, we are guaranteed by the converse of the Pythagorean Theorem that the triangle is a right triangle.

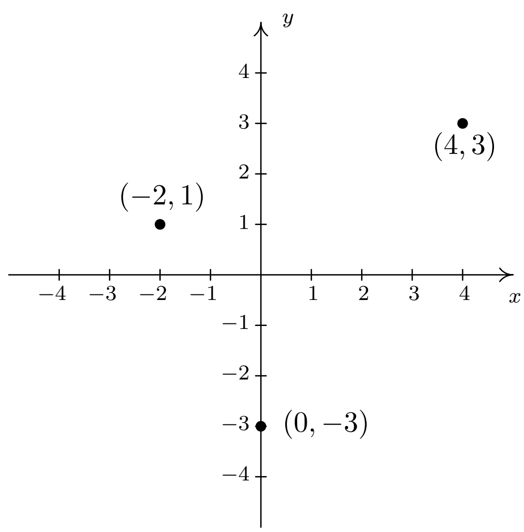
    (b) Show that $|AC|^2+|BC|^2=|AB|^2$

1.2 Relations

From one point of view,[1] all of Precalculus can be thought of as studying sets of points in the plane. With the Cartesian Plane now fresh in our memory we can discuss those sets in more detail and as usual, we begin with a definition.

Definition 1.4. A **relation** is a set of points in the plane.

Since relations are sets, we can describe them using the techniques presented in Section 1.1.1. That is, we can describe a relation verbally, using the roster method, or using set-builder notation. Since the elements in a relation are points in the plane, we often try to describe the relation graphically or algebraically as well. Depending on the situation, one method may be easier or more convenient to use than another. As an example, consider the relation $R = \{(-2,1),(4,3),(0,-3)\}$. As written, R is described using the roster method. Since R consists of points in the plane, we follow our instinct and plot the points. Doing so produces the **graph** of R.



The graph of R.

In the following example, we graph a variety of relations.

Example 1.2.1. Graph the following relations.

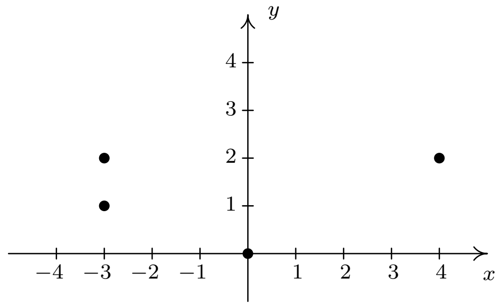
1. $A = \{(0,0),(-3,1),(4,2),(-3,2)\}$
2. $HLS_1 = \{(x,3) \mid -2 \leq x \leq 4\}$
3. $HLS_2 = \{(x,3) \mid -2 \leq x < 4\}$
4. $V = \{(3,y) \mid y \text{ is a real number}\}$
5. $H = \{(x,y) \mid y = -2\}$
6. $R = \{(x,y) \mid 1 < y \leq 3\}$

[1] Carl's, of course.

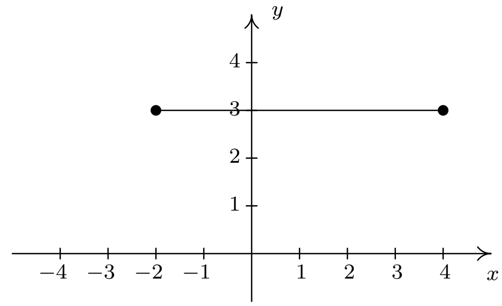
Solution.

1. To graph A, we simply plot all of the points which belong to A, as shown below on the left.

2. Don't let the notation in this part fool you. The name of this relation is HLS_1, just like the name of the relation in number 1 was A. The letters and numbers are just part of its name, just like the numbers and letters of the phrase 'King George III' were part of George's name. In words, $\{(x, 3) \mid -2 \leq x \leq 4\}$ reads 'the set of points $(x, 3)$ such that $-2 \leq x \leq 4$.' All of these points have the same y-coordinate, 3, but the x-coordinate is allowed to vary between -2 and 4, inclusive. Some of the points which belong to HLS_1 include some friendly points like: $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$, $(2, 3)$, $(3, 3)$, and $(4, 3)$. However, HLS_1 also contains the points $(0.829, 3)$, $\left(-\frac{5}{6}, 3\right)$, $(\sqrt{\pi}, 3)$, and so on. It is impossible[2] to list all of these points, which is why the variable x is used. Plotting several friendly representative points should convince you that HLS_1 describes the horizontal line segment from the point $(-2, 3)$ up to and including the point $(4, 3)$.



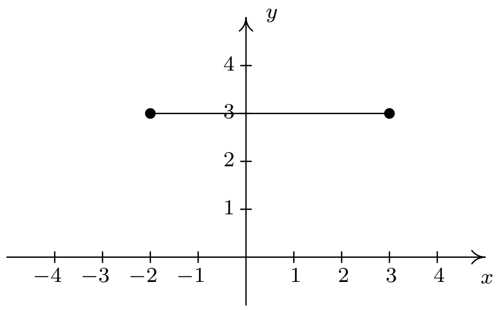
The graph of A



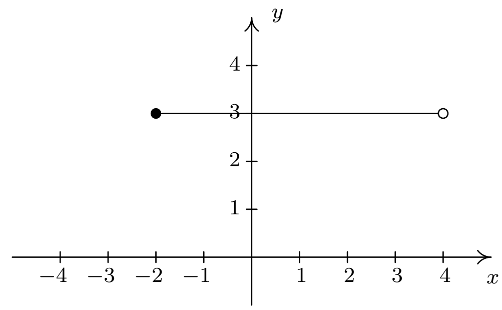
The graph of HLS_1

3. HLS_2 is hauntingly similar to HLS_1. In fact, the only difference between the two is that instead of '$-2 \leq x \leq 4$' we have '$-2 \leq x < 4$'. This means that we still get a horizontal line segment which includes $(-2, 3)$ and extends to $(4, 3)$, but we do *not* include $(4, 3)$ because of the strict inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2, 3)$ end at $(3, 3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1, 3)$, $(3.5, 3)$, $(3.9, 3)$, $(3.99, 3)$, and so forth. There is no real number that comes 'immediately before' 4, so to describe the set of points we want, we draw the horizontal line segment starting at $(-2, 3)$ and draw an open circle at $(4, 3)$ as depicted below on the right.

[2] Really impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.



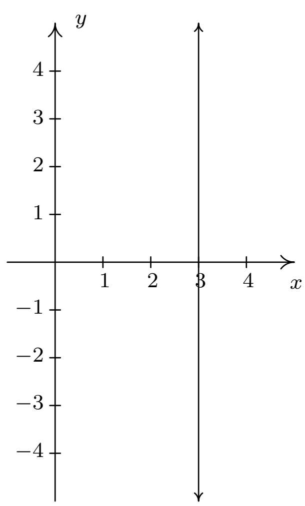
This is NOT the correct graph of HLS_2



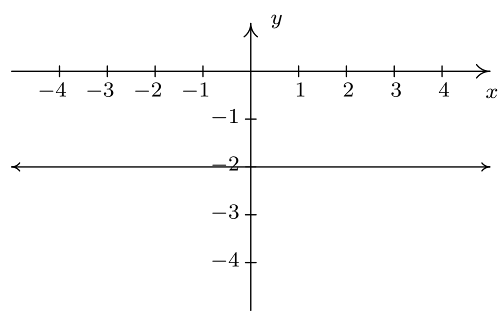
The graph of HLS_2

4. Next, we come to the relation V, described as the set of points $(3, y)$ such that y is a real number. All of these points have an x-coordinate of 3, but the y-coordinate is free to be whatever it wants to be, without restriction.[3] Plotting a few 'friendly' points of V should convince you that all the points of V lie on the vertical line[4] $x = 3$. Since there is no restriction on the y-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of V is below on the left.

5. Though written slightly differently, the relation $H = \{(x, y) \,|\, y = -2\}$ is similar to the relation V above in that only one of the coordinates, in this case the y-coordinate, is specified, leaving x to be 'free'. Plotting some representative points gives us the horizontal line $y = -2$.



The graph of V



The graph of H

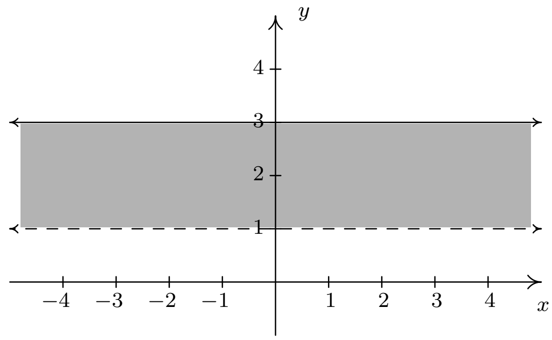
6. For our last example, we turn to $R = \{(x, y) \,|\, 1 < y \leq 3\}$. As in the previous example, x is free to be whatever it likes. The value of y, on the other hand, while not completely free, is permitted to roam between 1 and 3 excluding 1, but including 3. After plotting some[5] friendly elements of R, it should become clear that R consists of the region between the horizontal

[3] We'll revisit the concept of a 'free variable' in Section 8.1.

[4] Don't worry, we'll be refreshing your memory about vertical and horizontal lines in just a moment!

[5] The word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern.

lines $y = 1$ and $y = 3$. Since R requires that the y-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to R.



The graph of R □

The relations V and H in the previous example lead us to our final way to describe relations: **algebraically**. We can more succinctly describe the points in V as those points which satisfy the equation '$x = 3$'. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for x and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose x-coordinate is 3. Similarly, the relation H above can be described by the equation '$y = -2$'. At some point in your mathematical upbringing, you probably learned the following.

Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

Given that the very simple equations $x = a$ and $y = b$ produced lines, it's natural to wonder what shapes other equations might yield. Thus our next objective is to study the graphs of equations in a more general setting as we continue to unite Algebra and Geometry.

1.2.1 Graphs of Equations

In this section, we delve more deeply into the connection between Algebra and Geometry by focusing on graphing relations described by equations. The main idea of this section is the following.

The Fundamental Graphing Principle

The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Here, 'x and y satisfy the equation' means 'x and y make the equation true'. It is at this point that we gain some insight into the word 'relation'. If the equation to be graphed contains both x and y, then the equation itself is what is relating the two variables. More specifically, in the next two examples, we consider the graph of the equation $x^2 + y^3 = 1$. Even though it is not specifically spelled out, what we are doing is graphing the relation $R = \{(x,y) \,|\, x^2 + y^3 = 1\}$. The points (x,y) we graph belong to the *relation* R and are necessarily *related* by the equation $x^2 + y^3 = 1$, since it is those pairs of x and y which make the equation true.

Example 1.2.2. Determine whether or not $(2,-1)$ is on the graph of $x^2 + y^3 = 1$.

Solution. We substitute $x = 2$ and $y = -1$ into the equation to see if the equation is satisfied.

$$\begin{array}{rcl} (2)^2 + (-1)^3 & \stackrel{?}{=} & 1 \\ 3 & \neq & 1 \end{array}$$

Hence, $(2,-1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

Example 1.2.3. Graph $x^2 + y^3 = 1$.

Solution. To efficiently generate points on the graph of this equation, we first solve for y

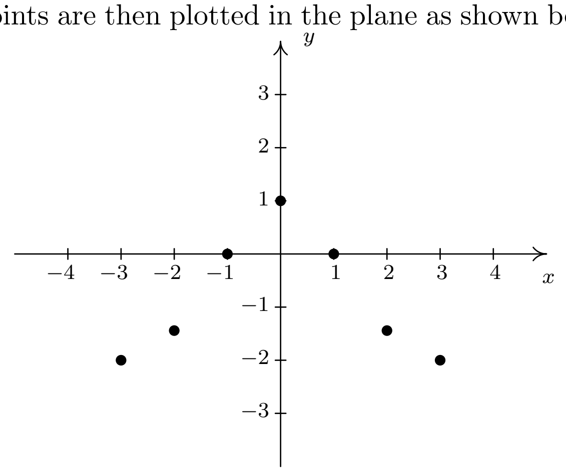
$$\begin{array}{rcl} x^2 + y^3 & = & 1 \\ y^3 & = & 1 - x^2 \\ \sqrt[3]{y^3} & = & \sqrt[3]{1 - x^2} \\ y & = & \sqrt[3]{1 - x^2} \end{array}$$

We now substitute a value in for x, determine the corresponding value y, and plot the resulting point (x,y). For example, substituting $x = -3$ into the equation yields

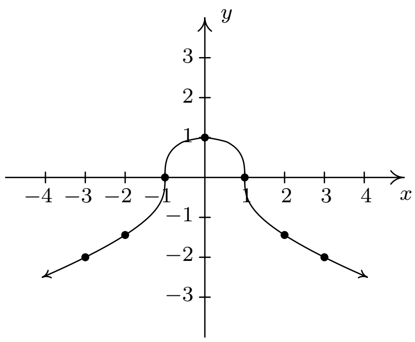
$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

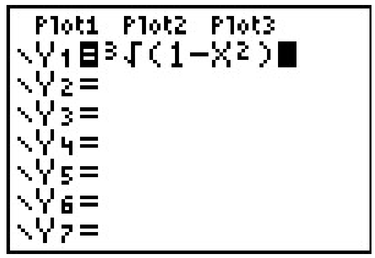
so the point $(-3,-2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown below.

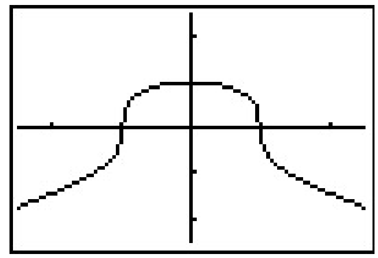
x	y	(x,y)
-3	-2	$(-3,-2)$
-2	$-\sqrt[3]{3}$	$(-2,-\sqrt[3]{3})$
-1	0	$(-1,0)$
0	1	$(0,1)$
1	0	$(1,0)$
2	$-\sqrt[3]{3}$	$(2,-\sqrt[3]{3})$
3	-2	$(3,-2)$



Remember, these points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots'. Doing so would result in a curve similar to the one pictured below on the far left.







Don't worry if you don't get all of the little bends and curves just right − Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can reach for a graphing calculator, input the formula as shown above, and graph. □

Of all of the points on the graph of an equation, the places where the graph crosses or touches the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

Definition 1.5. Suppose the graph of an equation is given.

- A point on a graph which is also on the x-axis is called an ***x*-intercept** of the graph.
- A point on a graph which is also on the y-axis is called an ***y*-intercept** of the graph.

In our previous example the graph had two x-intercepts, $(-1,0)$ and $(1,0)$, and one y-intercept, $(0,1)$. The graph of an equation can have any number of intercepts, including none at all! Since x-intercepts lie on the x-axis, we can find them by setting $y=0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x=0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

Finding the Intercepts of the Graph of an Equation

Given an equation involving x and y, we find the intercepts of the graph as follows:

- x-intercepts have the form $(x,0)$; set $y=0$ in the equation and solve for x.
- y-intercepts have the form $(0,y)$; set $x=0$ in the equation and solve for y.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the y-axis. To actually prove this analytically, we assume (x,y) is a generic point on the graph of the equation. That is, we assume $x^2+y^3=1$ is true. As we learned in Section 1.1, the point symmetric to (x,y) about the y-axis is $(-x,y)$. To show that the graph is symmetric about the y-axis, we need to show that $(-x,y)$ satisfies the equation $x^2+y^3=1$, too. Substituting $(-x,y)$ into the equation gives

$$\begin{aligned} (-x)^2+(y)^3 &\stackrel{?}{=} 1 \\ x^2+y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we are assuming the original equation $x^2+y^3=1$ is true, we have shown that $(-x,y)$ satisfies the equation (since it leads to a true result) and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.1. We summarize the procedure in the following result.

Testing the Graph of an Equation for Symmetry

To test the graph of an equation for symmetry

- about the y-axis – substitute $(-x,y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the x-axis – substitute $(x,-y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the origin - substitute $(-x,-y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as demonstrated in the next example.

Example 1.2.4. Find the x- and y-intercepts (if any) of the graph of $(x-2)^2+y^2=1$. Test for symmetry. Plot additional points as needed to complete the graph.

Solution. To look for x-intercepts, we set $y=0$ and solve

$$\begin{aligned} (x-2)^2+y^2 &= 1 \\ (x-2)^2+0^2 &= 1 \\ (x-2)^2 &= 1 \\ \sqrt{(x-2)^2} &= \sqrt{1} \quad \text{extract square roots} \\ x-2 &= \pm 1 \\ x &= 2\pm 1 \\ x &= 3,1 \end{aligned}$$

We get two answers for x which correspond to two x-intercepts: $(1,0)$ and $(3,0)$. Turning our attention to y-intercepts, we set $x=0$ and solve