

Solving an Absolute Value Equation

Next, we will learn how to solve an absolute value equation. To solve an equation such as $|2x - 6| = 8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8 . This leads to two different equations we can solve independently.

$$\begin{array}{rcl} 2x - 6 = 8 & \text{or} & 2x - 6 = -8 \\ 2x = 14 & & 2x = -2 \\ x = 7 & & x = -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

absolute value equations

The absolute value of x is written as $|x|$. It has the following properties:

$$\text{If } x \geq 0, \text{ then } |x| = x.$$

$$\text{If } x < 0, \text{ then } |x| = -x.$$

For real numbers A and B , an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

An absolute value equation in the form $|ax + b| = c$ has the following properties:

$$\text{If } c < 0, |ax + b| = c \text{ has no solution.}$$

$$\text{If } c = 0, |ax + b| = c \text{ has one solution.}$$

$$\text{If } c > 0, |ax + b| = c \text{ has two solutions.}$$

How To...

Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$, write and solve two equations: $ax + b = c$ and $ax + b = -c$.

Example 8 Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x + 4| = 8$

b. $|3x + 4| = -9$

c. $|3x - 5| - 4 = 6$

d. $|-5x + 10| = 0$

Solution

a. $|6x + 4| = 8$

Write two equations and solve each:

$$6x + 4 = 8$$

$$6x + 4 = -8$$

$$6x = 4$$

$$6x = -12$$

$$x = \frac{2}{3}$$

$$x = -2$$

The two solutions are $\frac{2}{3}$ and -2 .

b. $|3x + 4| = -9$

There is no solution as an absolute value cannot be negative.

LEARNING OBJECTIVES

In this section you will:

- Use interval notation.
- Solve linear inequalities.
- Solve compound linear inequalities of both "and" and "or" type.
- Solve absolute value inequalities.

2.7 LINEAR INEQUALITIES AND ABSOLUTE VALUE INEQUALITIES



Figure 1

It is not easy to make the honor roll at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in **Figure 2**. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

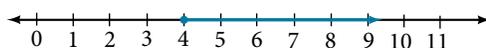


Figure 2

We can use set-builder notation: $\{x|x \geq 4\}$, which translates to “all real numbers x such that x is greater than or equal to 4.” Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be “equaled.” A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6 , including -2 , but not including 6 ; $(-1, 0)$, all real numbers between, but not including -1 and 0 ; and $(-\infty, 1]$, all real numbers less than and including 1 . **Table 1** outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b , but not including a or b	$\{x a < x < b\}$	(a, b)
All real numbers greater than a , but not including a	$\{x x > a\}$	(a, ∞)
All real numbers less than b , but not including b	$\{x x < b\}$	$(-\infty, b)$
All real numbers greater than a , including a	$\{x x \geq a\}$	$[a, \infty)$

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers less than b , including b	$\{x \mid x \leq b\}$	$(-\infty, b]$
All real numbers between a and b , including a	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between a and b , including b	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between a and b , including a and b	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x \mid x < a \text{ and } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Table 1

Example 1 Using Interval Notation to Express All Real Numbers Greater Than or Equal to a

Use interval notation to indicate all real numbers greater than or equal to -2 .

Solution Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Try It #1

Use interval notation to indicate all real numbers between and including -3 and 5 .

Example 2 Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1 .

Solution We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1 . So, this interval begins at $-\infty$ and ends at -1 , which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1 , which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, \cup , between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

Try It #2

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

properties of inequalities**Addition Property**

If $a < b$, then $a + c < b + c$.

Multiplication Property

If $a < b$ and $c > 0$, then $ac < bc$.

If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

Example 3 Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

a. $x - 15 < 4$

b. $6 \geq x - 1$

c. $x + 7 > 9$

Solution The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

- a.** $x - 15 < 4$
 $x - 15 + 15 < 4 + 15$ Add 15 to both sides.
 $x < 19$
- b.** $6 \geq x - 1$
 $6 + 1 \geq x - 1 + 1$ Add 1 to both sides.
 $7 \geq x$
- c.** $x + 7 > 9$
 $x + 7 - 7 > 9 - 7$ Subtract 7 from both sides.
 $x > 2$

Try It #3

Solve: $3x - 2 < 1$.

Example 4 Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

- a.** $3x < 6$ **b.** $-2x - 1 \geq 5$ **c.** $5 - x > 10$

Solution

- a.** $3x < 6$
 $\frac{1}{3}(3x) < (6)\frac{1}{3}$
 $x < 2$
- b.** $-2x - 1 \geq 5$
 $-2x \geq 6$
 $\left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right)$ Multiply by $-\frac{1}{2}$.
 $x \leq -3$ Reverse the inequality.
- c.** $5 - x > 10$
 $-x > 5$
 $(-1)(-x) > (5)(-1)$ Multiply by -1 .
 $x < -5$ Reverse the inequality.

Try It #4

Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 5 Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution Solving this inequality is similar to solving an equation up until the last step.

$$\begin{aligned} 13 - 7x &\geq 10x - 4 \\ 13 - 17x &\geq -4 && \text{Move variable terms to one side of the inequality.} \\ -17x &\geq -17 && \text{Isolate the variable term.} \\ x &\leq 1 && \text{Dividing both sides by } -17 \text{ reverses the inequality.} \end{aligned}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

Try It #5

Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

Example 6 Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

Solution We begin solving in the same way we do when solving an equation.

$$\begin{aligned} -\frac{3}{4}x &\geq -\frac{5}{8} + \frac{2}{3}x \\ -\frac{3}{4}x - \frac{2}{3}x &\geq -\frac{5}{8} && \text{Put variable terms on one side.} \\ -\frac{9}{12}x - \frac{8}{12}x &\geq -\frac{5}{8} && \text{Write fractions with common denominator.} \\ -\frac{17}{12}x &\geq -\frac{5}{8} \\ x &\leq -\frac{5}{8} \left(-\frac{12}{17} \right) && \text{Multiplying by a negative number reverses the inequality.} \\ x &\leq \frac{15}{34} \end{aligned}$$

The solution set is the interval $(-\infty, \frac{15}{34}]$.

Try It #6

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example 7 Solving a Compound Inequality

Solve the compound inequality: $3 \leq 2x + 2 < 6$.

Solution The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$\begin{aligned} 3 &\leq 2x + 2 && \text{and} && 2x + 2 < 6 \\ 1 &\leq 2x && && 2x < 4 \\ \frac{1}{2} &\leq x && && x < 2 \end{aligned}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$\begin{array}{rcl} 3 \leq 2x + 2 < 6 & & \\ 1 \leq 2x < 4 & \text{Isolate the variable term, and subtract 2 from all three parts.} & \\ \frac{1}{2} \leq x < 2 & \text{Divide through all three parts by 2.} & \end{array}$$

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

Try It #7

Solve the compound inequality: $4 < 2x - 8 \leq 10$.

Example 8 Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution Let's try the first method. Write two inequalities:

$$\begin{array}{rcl} 3 + x > 7x - 2 & \text{and} & 7x - 2 > 5x - 10 \\ 3 > 6x - 2 & & 2x - 2 > -10 \\ 5 > 6x & & 2x > -8 \\ \frac{5}{6} > x & & x > -4 \\ x < \frac{5}{6} & & -4 < x \end{array}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See **Figure 3**.



Figure 3

Try It #8

Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x , as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, |A| \leq B, |A| > B, \text{ or } |A| \geq B,$$

Where A , and sometimes B , represents an algebraic expression dependent on a variable x . Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of x -values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x - 600|$, and therefore, $|x - 600| \leq 200$ or

$$\begin{aligned} -200 &\leq x - 600 \leq 200 \\ -200 + 600 &\leq x - 600 + 600 \leq 200 + 600 \\ 400 &\leq x \leq 800 \end{aligned}$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

absolute value inequalities

For an algebraic expression X , and $k > 0$, an **absolute value inequality** is an inequality of the form

$$|X| < k \text{ is equivalent to } -k < X < k$$

$$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

Example 9 Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in **Figure 4**, to represent the condition to be satisfied.

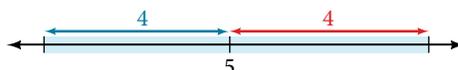


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x - 5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x - 5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{aligned} x - 5 &\leq 4 & \text{and} & & x - 5 &\geq -4 \\ x &\leq 9 & & & x &\geq 1 \end{aligned}$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x - 5| \leq 4$ is equivalent to $[1, 9]$ in interval notation.

Try It #9

Describe all x -values within a distance of 3 from the number 2.

Example 10 Solving an Absolute Value Inequality

Solve $|x - 1| \leq 3$.

Solution

$$\begin{aligned} |x - 1| &\leq 3 \\ -3 &\leq x - 1 \leq 3 \\ -2 &\leq x \leq 4 \\ [-2, 4] \end{aligned}$$