How To…

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

Example 2 **Plotting a Complex Number on the Complex Plane**

Plot the complex number $3 - 4i$ on the complex plane.

Solution The real part of the complex number is 3, and the imaginary part is -4. We plot the ordered pair $(3, -4)$ as shown in **Figure 4**.

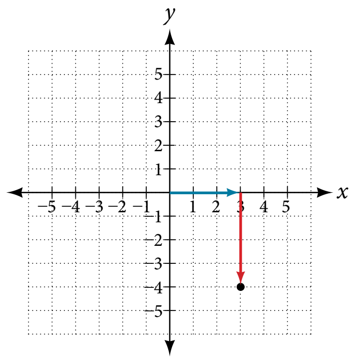

Figure 4

Try It #2

Plot the complex number $-4 - i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and then combine the imaginary parts.

complex numbers: addition and subtraction

Adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtracting complex numbers:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

How To…

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

Example 3 **Adding and Subtracting Complex Numbers**

Add or subtract as indicated.

a. $(3 - 4i) + (2 + 5i)$ **b.** $(-5 + 7i) - (-11 + 2i)$

Solution We add the real parts and add the imaginary parts.

a. $$\begin{aligned}(3-4i)+(2+5i) &= 3-4i+2+5i\\ &= 3+2+(-4i)+5i\\ &= (3+2)+(-4+5)i\\ &= 5+i\end{aligned}$$

b. $$\begin{aligned}(-5+7i)-(-11+2i) &= -5+7i+11-2i\\ &= -5+11+7i-2i\\ &= (-5+11)+(7-2)i\\ &= 6+5i\end{aligned}$$

Try It #3

Subtract $2+5i$ from $3-4i$.

Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

Multiplying a Complex Number by a Real Number

Lets begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. Consider, for example, $3(6+2i)$:

$$\begin{aligned}3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.}\\ &= 18+6i && \text{Simplify.}\end{aligned}$$

How To…

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

Example 4 Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

Solution Distribute the 4.

$$\begin{aligned}4(2+5i) &= (4\cdot 2)+(4\cdot 5i)\\ &= 8+20i\end{aligned}$$

Try It #4

Find the product: $\frac{1}{2}(5-2i)$.

Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or more specifically the FOIL method because we are dealing with binomials. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. The difference with complex numbers is that when we get a squared term, i^2, it equals -1.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci-bd && i^2=-1\\ &= (ac-bd)+(ad+bc)i && \text{Group real terms and imaginary terms.}\end{aligned}$$

How To...

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Remember that $i^2 = -1$.
3. Group together the real terms and the imaginary terms

Example 5 **Multiplying a Complex Number by a Complex Number**

Multiply: $(4 + 3i)(2 - 5i)$.

Solution

$$\begin{aligned}(4 + 3i)(2 - 5i) &= 4(2) - 4(5i) + 3i(2) - (3i)(5i)\\ &= 8 - 20i + 6i - 15(i^2)\\ &= (8 + 15) + (-20 + 6)i\\ &= 23 - 14i\end{aligned}$$

Try It #5

Multiply: $(3 - 4i)(2 + 3i)$.

Dividing Complex Numbers

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form $a + bi$. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a + bi$ is $a - bi$. For example, the product of $a + bi$ and $a - bi$ is

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 + b^2\end{aligned}$$

The result is a real number.

Note that complex conjugates have an opposite relationship: The complex conjugate of $a + bi$ is $a - bi$, and the complex conjugate of $a - bi$ is $a + bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c + di$ by $a + bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c + di}{a + bi} \text{ where } a \neq 0 \text{ and } b \neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c + di)}{(a + bi)} \cdot \frac{(a - bi)}{(a - bi)} = \frac{(c + di)(a - bi)}{(a + bi)(a - bi)}$$

Apply the distributive property.

$$= \frac{ca - cbi + adi - bdi^2}{a^2 - abi + abi - b^2 i^2}$$

Simplify, remembering that $i^2 = -1$.

$$= \frac{ca - cbi + adi - bd(-1)}{a^2 - abi + abi - b^2(-1)}$$

$$= \frac{(ca + bd) + (ad - cb)i}{a^2 + b^2}$$

the complex conjugate

The **complex conjugate** of a complex number $a + bi$ is $a - bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

Example 6 Finding Complex Conjugates

Find the complex conjugate of each number.

a. $2 + i\sqrt{5}$ **b.** $-\frac{1}{2}i$

Solution

a. The number is already in the form $a + bi$. The complex conjugate is $a - bi$, or $2 - i\sqrt{5}$.

b. We can rewrite this number in the form $a + bi$ as $0 - \frac{1}{2}i$. The complex conjugate is $a - bi$, or $0 + \frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

Analysis *Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by i.*

Try It #6

Find the complex conjugate of $-3 + 4i$.

How To…

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

Example 7 Dividing Complex Numbers

Divide: $(2 + 5i)$ by $(4 - i)$.

Solution We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (using FOIL).

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} = \frac{8+2i+20i+5i^2}{16+4i-4i-i^2}$$

$$= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} \quad \text{Because } i^2 = -1.$$

$$= \frac{3+22i}{17}$$

$$= \frac{3}{17} + \frac{22}{17}i \quad \text{Separate real and imaginary parts.}$$

Note that this expresses the quotient in standard form.

Simplifying Powers of *i*

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$i^1 = \mathrm{i}$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$$
$$i^5 = i^4 \cdot i = 1 \cdot i = \mathrm{i}$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by increasing powers, we will see a cycle of four. Let's examine the next four powers of i.

$$i^6 = i^5 \cdot i = i \cdot i = i^2 = -1$$
$$i^7 = i^6 \cdot i = i^2 \cdot i = i^3 = -i$$
$$i^8 = i^7 \cdot i = i^3 \cdot i = i^4 = 1$$
$$i^9 = i^8 \cdot i = i^4 \cdot i = i^5 = \mathrm{i}$$

The cycle is repeated continuously: $i, -1, -i, 1$, every four powers.

Example 8 Simplifying Powers of *i*

Evaluate: i^{35}.

Solution Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$.

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = (i^4)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

Try It #7

Evaluate: i^{18}

Q & A…

Can we write i^{35} in other helpful ways?

As we saw in **Example 8**, we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. **Table 1** shows some other possible factorizations.

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
Reduced form	$(i^2)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot (i^4)^4$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Table 1

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

Access these online resources for additional instruction and practice with complex numbers.

- Adding and Subtracting Complex Numbers (http://openstaxcollege.org/l/addsubcomplex)
- Multiply Complex Numbers (http://openstaxcollege.org/l/multiplycomplex)
- Multiplying Complex Conjugates (http://openstaxcollege.org/l/multcompconj)
- Raising *i* to Powers (http://openstaxcollege.org/l/raisingi)