

## LEARNING OBJECTIVES

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In this section you will:

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by using the quadratic formula.

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## 2.5 QUADRATIC EQUATIONS

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Figure 1

The computer monitor on the left in **Figure 1** is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

### Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as  $2x^2 + 3x - 1 = 0$  and  $x^2 - 4 = 0$  are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ , where  $a$  and  $b$  are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression  $(x - 2)(x + 3)$  by multiplying the two factors together.

$$\begin{aligned}(x - 2)(x + 3) &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

The product is a quadratic expression. Set equal to zero,  $x^2 + x - 6 = 0$  is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form,  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers, and  $a \neq 0$ . The equation  $x^2 + x - 6 = 0$  is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

***the zero-product property and quadratic equations***

The **zero-product property** states

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0,$$

where  $a$  and  $b$  are real numbers or algebraic expressions.

A **quadratic equation** is an equation containing a second-degree polynomial; for example

$$ax^2 + bx + c = 0$$

where  $a$ ,  $b$ , and  $c$  are real numbers, and if  $a \neq 0$ , it is in standard form.

**Solving Quadratics with a Leading Coefficient of 1**

In the quadratic equation  $x^2 + x - 6 = 0$ , the leading coefficient, or the coefficient of  $x^2$ , is 1. We have one method of factoring quadratic equations in this form.

*How To...*

Given a quadratic equation with the leading coefficient of 1, factor it.

1. Find two numbers whose product equals  $c$  and whose sum equals  $b$ .
2. Use those numbers to write two factors of the form  $(x + k)$  or  $(x - k)$ , where  $k$  is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and  $-2$ , the factors are  $(x + 1)(x - 2)$ .
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

**Example 1 Factoring and Solving a Quadratic with Leading Coefficient of 1**

Factor and solve the equation:  $x^2 + x - 6 = 0$ .

**Solution** To factor  $x^2 + x - 6 = 0$ , we look for two numbers whose product equals  $-6$  and whose sum equals 1. Begin by looking at the possible factors of  $-6$ .

$$\begin{aligned} 1 \cdot (-6) \\ (-6) \cdot 1 \\ 2 \cdot (-3) \\ 3 \cdot (-2) \end{aligned}$$

The last pair,  $3 \cdot (-2)$  sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ (x - 2) &= 0 \\ x &= 2 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The two solutions are 2 and  $-3$ . We can see how the solutions relate to the graph in **Figure 2**. The solutions are the  $x$ -intercepts of  $x^2 + x - 6 = 0$ .

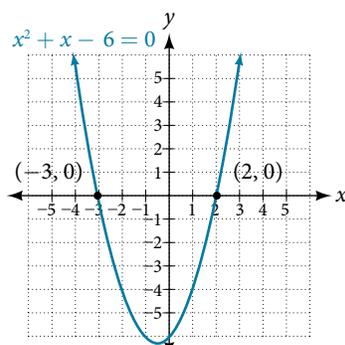


Figure 2

*Try It #1*

Factor and solve the quadratic equation:  $x^2 - 5x - 6 = 0$ .

**Example 2** Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring:  $x^2 + 8x + 15 = 0$ .

**Solution** Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$$\begin{aligned} &1 \cdot 15 \\ &3 \cdot 5 \\ &(-1) \cdot (-15) \\ &(-3) \cdot (-5) \end{aligned}$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x + 3)(x + 5) &= 0 \\ (x + 3) &= 0 \\ x &= -3 \\ (x + 5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are  $-3$  and  $-5$ .

*Try It #2*

Solve the quadratic equation by factoring:  $x^2 - 4x - 21 = 0$ .

**Example 3** Using the Zero-Product Property to Solve a Quadratic Equation Written as the Difference of Squares

Solve the difference of squares equation using the zero-product property:  $x^2 - 9 = 0$ .

**Solution** Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x - 3)(x + 3) &= 0 \\ (x - 3) &= 0 \\ x &= 3 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and  $-3$ .

*Try It #3*

Solve by factoring:  $x^2 - 25 = 0$ .

**Factoring and Solving a Quadratic Equation of Higher Order**

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms. With the equation in standard form, let's review the grouping procedures:

1. With the quadratic in standard form,  $ax^2 + bx + c = 0$ , multiply  $a \cdot c$ .
2. Find two numbers whose product equals  $ac$  and whose sum equals  $b$ .
3. Rewrite the equation replacing the  $bx$  term with two terms using the numbers found in step 1 as coefficients of  $x$ .
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

**Example 4 Solving a Quadratic Equation Using Grouping**

Use grouping to factor and solve the quadratic equation:  $4x^2 + 15x + 9 = 0$ .

**Solution** First, multiply  $ac$ :  $4(9) = 36$ . Then list the factors of 36.

$$\begin{aligned} &1 \cdot 36 \\ &2 \cdot 18 \\ &3 \cdot 12 \\ &4 \cdot 9 \\ &6 \cdot 6 \end{aligned}$$

The only pair of factors that sums to 15 is  $3 + 12$ . Rewrite the equation replacing the  $b$  term,  $15x$ , with two terms using 3 and 12 as coefficients of  $x$ . Factor the first two terms, and then factor the last two terms.

$$\begin{aligned} 4x^2 + 3x + 12x + 9 &= 0 \\ x(4x + 3) + 3(4x + 3) &= 0 \\ (4x + 3)(x + 3) &= 0 \end{aligned}$$

Solve using the zero-product property.

$$\begin{aligned} (4x + 3)(x + 3) &= 0 \\ (4x + 3) &= 0 \\ x &= -\frac{3}{4} \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are  $-\frac{3}{4}$  and  $-3$ . See **Figure 3**.

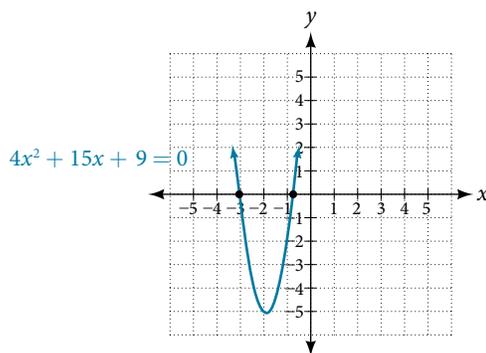


Figure 3

**Try It #4**

Solve using factoring by grouping:  $12x^2 + 11x + 2 = 0$ .

**Example 5 Solving a Higher Degree Quadratic Equation by Factoring**

Solve the equation by factoring:  $-3x^3 - 5x^2 - 2x = 0$ .

**Solution** This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out  $-x$  from all of the terms and then proceed with grouping.

$$\begin{aligned} -3x^3 - 5x^2 - 2x &= 0 \\ -x(3x^2 + 5x + 2) &= 0 \end{aligned}$$

Use grouping on the expression in parentheses.

$$\begin{aligned} -x(3x^2 + 3x + 2x + 2) &= 0 \\ -x[3x(x + 1) + 2(x + 1)] &= 0 \\ -x(3x + 2)(x + 1) &= 0 \end{aligned}$$

Now, we use the zero-product property. Notice that we have three factors.

$$\begin{aligned} -x &= 0 \\ x &= 0 \\ 3x + 2 &= 0 \\ x &= -\frac{2}{3} \\ x + 1 &= 0 \\ x &= -1 \end{aligned}$$

The solutions are 0,  $-\frac{2}{3}$ , and  $-1$ .

### Try It #5

Solve by factoring:  $x^3 + 11x^2 + 10x = 0$ .

## Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the  $x^2$  term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the  $x^2$  term so that the square root property can be used.

### *the square root property*

With the  $x^2$  term isolated, the square root property states that:

$$\text{if } x^2 = k, \text{ then } x = \pm \sqrt{k}$$

where  $k$  is a nonzero real number.

### How To...

Given a quadratic equation with an  $x^2$  term but no  $x$  term, use the square root property to solve it.

1. Isolate the  $x^2$  term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a  $\pm$  sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the  $\pm$  sign.

### Example 6 Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property:  $x^2 = 8$ .

**Solution** Take the square root of both sides, and then simplify the radical. Remember to use a  $\pm$  sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm \sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are  $2\sqrt{2}$  and  $-2\sqrt{2}$ .

### Example 7 Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation:  $4x^2 + 1 = 7$ .

**Solution** First, isolate the  $x^2$  term. Then take the square root of both sides.

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm \frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are  $\frac{\sqrt{6}}{2}$  and  $-\frac{\sqrt{6}}{2}$ .

*Try It #6*

Solve the quadratic equation using the square root property:  $3(x - 4)^2 = 15$ .

**Completing the Square**

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient,  $a$ , must equal 1. If it does not, then divide the entire equation by  $a$ . Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example  $x^2 + 4x + 1 = 0$  to illustrate each step.

1. Given a quadratic equation that cannot be factored, and with  $a = 1$ , first add or subtract the constant term to the right sign of the equal sign.

$$x^2 + 4x = -1$$

2. Multiply the  $b$  term by  $\frac{1}{2}$  and square it.

$$\frac{1}{2}(4) = 2$$

$$2^2 = 4$$

3. Add  $\left(\frac{1}{2}b\right)^2$  to both sides of the equal sign and simplify the right side. We have

$$x^2 + 4x + 4 = -1 + 4$$

$$x^2 + 4x + 4 = 3$$

4. The left side of the equation can now be factored as a perfect square.

$$x^2 + 4x + 4 = 3$$

$$(x + 2)^2 = 3$$

5. Use the square root property and solve.

$$\sqrt{(x + 2)^2} = \pm \sqrt{3}$$

$$x + 2 = \pm \sqrt{3}$$

$$x = -2 \pm \sqrt{3}$$

6. The solutions are  $-2 + \sqrt{3}$  and  $-2 - \sqrt{3}$ .

**Example 8 Solving a Quadratic by Completing the Square**

Solve the quadratic equation by completing the square:  $x^2 - 3x - 5 = 0$ .

**Solution** First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5$$

Then, take  $\frac{1}{2}$  of the  $b$  term and square it.

$$\frac{1}{2}(-3) = -\frac{3}{2}$$

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

Add the result to both sides of the equal sign.

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2$$

$$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

Use the square root property and solve.

$$\begin{aligned}\sqrt{\left(x - \frac{3}{2}\right)^2} &= \pm \sqrt{\frac{29}{4}} \\ \left(x - \frac{3}{2}\right) &= \pm \frac{\sqrt{29}}{2} \\ x &= \frac{3}{2} \pm \frac{\sqrt{29}}{2}\end{aligned}$$

The solutions are  $\frac{3 + \sqrt{29}}{2}$  and  $\frac{3 - \sqrt{29}}{2}$ .

*Try It #7*

Solve by completing the square:  $x^2 - 6x = 13$ .

### Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by  $-1$  and obtain a positive  $a$ . Given  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:

$$ax^2 + bx = -c$$

2. As we want the leading coefficient to equal 1, divide through by  $a$ :

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Then, find  $\frac{1}{2}$  of the middle term, and add  $\left(\frac{1b}{2a}\right)^2 = \frac{b^2}{4a^2}$  to both sides of the equal sign:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Now, use the square root property, which gives

$$\begin{aligned}x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

6. Finally, add  $-\frac{b}{2a}$  to both sides of the equation and combine the terms on the right side. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

#### ***the quadratic formula***

Written in standard form,  $ax^2 + bx + c = 0$ , any quadratic equation can be solved using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

*How To...*

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form:  $ax^2 + bx + c = 0$ .
2. Make note of the values of the coefficients and constant term,  $a$ ,  $b$ , and  $c$ .
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

**Example 9** Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation:  $x^2 + 5x + 1 = 0$ .

**Solution** Identify the coefficients:  $a = 1$ ,  $b = 5$ ,  $c = 1$ . Then use the quadratic formula.

$$\begin{aligned} x &= \frac{-(-5) \pm \sqrt{(5)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 4}}{2} \\ &= \frac{-5 \pm \sqrt{21}}{2} \end{aligned}$$

**Example 10** Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve  $x^2 + x + 2 = 0$ .

**Solution** First, we identify the coefficients:  $a = 1$ ,  $b = 1$ , and  $c = 2$ .

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(1)^2 - (4) \cdot (1) \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are  $\frac{-1 + i\sqrt{7}}{2}$  and  $\frac{-1 - i\sqrt{7}}{2}$ .

*Try It #8*

Solve the quadratic equation using the quadratic formula:  $9x^2 + 3x - 2 = 0$ .

**The Discriminant**

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical,  $b^2 - 4ac$ . The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. **Table 1** relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2 - 4ac = 0$	One rational solution (double solution)
$b^2 - 4ac > 0$ , perfect square	Two rational solutions
$b^2 - 4ac > 0$ , not a perfect square	Two irrational solutions
$b^2 - 4ac < 0$	Two complex solutions

Table 1

**the discriminant**

For  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers, the **discriminant** is the expression under the radical in the quadratic formula:  $b^2 - 4ac$ . It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

**Example 11 Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation**

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a.  $x^2 + 4x + 4 = 0$     b.  $8x^2 + 14x + 3 = 0$     c.  $3x^2 - 5x - 2 = 0$     d.  $3x^2 - 10x + 15 = 0$

**Solution** Calculate the discriminant  $b^2 - 4ac$  for each equation and state the expected type of solutions.

a.  $x^2 + 4x + 4 = 0$

$$b^2 - 4ac = (4)^2 - 4(1)(4) = 0. \text{ There will be one rational double solution.}$$

b.  $8x^2 + 14x + 3 = 0$

$$b^2 - 4ac = (14)^2 - 4(8)(3) = 100. \text{ As 100 is a perfect square, there will be two rational solutions.}$$

c.  $3x^2 - 5x - 2 = 0$

$$b^2 - 4ac = (-5)^2 - 4(3)(-2) = 49. \text{ As 49 is a perfect square, there will be two rational solutions.}$$

d.  $3x^2 - 10x + 15 = 0$

$$b^2 - 4ac = (-10)^2 - 4(3)(15) = -80. \text{ There will be two complex solutions.}$$

**Using the Pythagorean Theorem**

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as  $a^2 + b^2 = c^2$ , where  $a$  and  $b$  refer to the legs of a right triangle adjacent to the  $90^\circ$  angle, and  $c$  refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

$$a^2 + b^2 = c^2$$

where  $a$  and  $b$  refer to the legs of a right triangle adjacent to the  $90^\circ$  angle, and  $c$  refers to the hypotenuse, as shown in **Figure 4**.

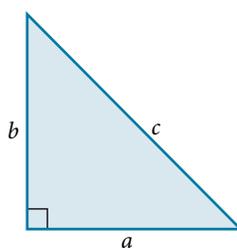


Figure 4