

LEARNING OBJECTIVES

In this section you will:

- Solve linear equations in one variable.
- Solve rational equations in one variable.
- Find the slope of a line that contains two given points.
- Graph linear equations.
- Use the point-slope and/or slope intercept formula to write the equation of a line that satisfies certain given properties.
- Given the equations of two lines, determine whether their graphs are parallel, perpendicular, or neither.
- Find the equation of a line parallel or perpendicular to a given line.

2.2 LINEAR EQUATIONS IN ONE VARIABLE

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays \$15.00/hr, and she opened a savings account with an initial deposit of \$400 on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately \$2,500, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in **Figure 1**.

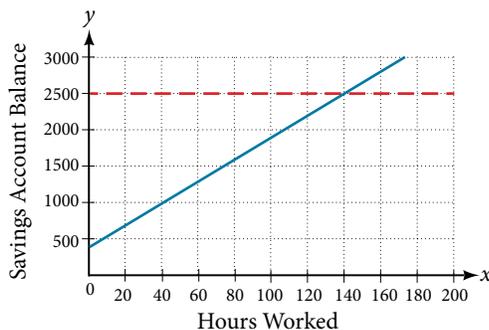


Figure 1

Solving Linear Equations in One Variable

A **linear equation** is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax + b = 0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent. An **identity equation** is true for all values of the variable. Here is an example of an identity equation.

$$3x = 2x + x$$

The **solution set** consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$5x + 2 = 3x - 6$$

$$2x = -8$$

$$x = -4$$

The solution set consists of one number: $\{-4\}$. It is the only solution and, therefore, we have solved a conditional equation.

An **inconsistent equation** results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$5x - 15 = 5x - 20$$

$$5x - 15 - 5x = 5x - 20 - 5x \quad \text{Subtract } 5x \text{ from both sides.}$$

$$-15 \neq -20$$

False statement

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

linear equation in one variable

A linear equation in one variable can be written in the form

$$ax + b = 0$$

where a and b are real numbers, $a \neq 0$.

How To...

Given a linear equation in one variable, use algebra to solve it.

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x = \text{_____}$, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b + c) = ab + ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example 1 Solving an Equation in One Variable

Solve the following equation: $2x + 7 = 19$.

Solution This equation can be written in the form $ax + b = 0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$2x + 7 = 19$$

$$2x = 12$$

$$x = 6$$

Subtract 7 from both sides.

Multiply both sides by $\frac{1}{2}$ or divide by 2.

The solution is 6.

Try It #1

Solve the linear equation in one variable: $2x + 1 = -9$.

Example 2 Solving an Equation Algebraically When the Variable Appears on Both Sides

Solve the following equation: $4(x - 3) + 12 = 15 - 5(x + 6)$.

Solution Apply standard algebraic properties.

$$4(x - 3) + 12 = 15 - 5(x + 6)$$

$$4x - 12 + 12 = 15 - 5x - 30$$

$$4x = -15 - 5x$$

$$9x = -15$$

$$x = -\frac{15}{9}$$

$$x = -\frac{5}{3}$$

Apply the distributive property.

Combine like terms.

Place x - terms on one side and simplify.

Multiply both sides by $\frac{1}{9}$, the reciprocal of 9.

Analysis This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x = -\frac{5}{3}$.

Try It #2

Solve the equation in one variable: $-2(3x - 1) + x = 14 - x$.

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is considered a **rational equation**.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}, \frac{1}{x-3}, \text{ or } \frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

Example 3 Solving a Rational Equation

Solve the rational equation: $\frac{7}{2x} - \frac{5}{3x} = \frac{22}{3}$.

Solution We have three denominators; $2x$, $3x$, and 3 . The LCD must contain $2x$, $3x$, and 3 . An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$(6x)\left(\frac{7}{2x} - \frac{5}{3x}\right) = \left(\frac{22}{3}\right)(6x)$$

$$(6x)\left(\frac{7}{2x}\right) - (6x)\left(\frac{5}{3x}\right) = \left(\frac{22}{3}\right)(6x) \quad \text{Use the distributive property.}$$

$$(6\cancel{x})\left(\frac{7}{2\cancel{x}}\right) - (6\cancel{x})\left(\frac{5}{3\cancel{x}}\right) = \left(\frac{22}{3}\right)(6x) \quad \text{Cancel out the common factors.}$$

$$3(7) - 2(5) = 22(2x) \quad \text{Multiply remaining factors by each numerator.}$$

$$21 - 10 = 44x$$

$$11 = 44x$$

$$\frac{11}{44} = x$$

$$\frac{1}{4} = x$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x + 1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x , $x - 1$, and $3x - 3$. First, factor all denominators. We then have x , $(x - 1)$, and $3(x - 1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x - 1)$. The x in the first denominator is separate from the x in the $(x - 1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x , one factor of $(x - 1)$, and the 3 . Thus, the LCD is the following:

$$x(x - 1)3 = 3x(x - 1)$$

So, both sides of the equation would be multiplied by $3x(x - 1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and $x^2 + 2x$. Once the second denominator is factored as $x^2 + 2x = x(x + 2)$, there is a common factor of x in both denominators and the LCD is $x(x + 2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b} = \frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} \times \frac{d}{d} = \frac{c}{d} \times \frac{b}{b}.$$

Multiply $a(d)$ and $b(c)$, which results in $ad = bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

rational equations

A **rational equation** contains at least one rational expression where the variable appears in at least one of the denominators.

How To...

Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.

Example 4 Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

Solution We have three denominators: x , 2 , and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0 .

Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x\left(\frac{2}{x} - \frac{3}{2}\right) &= \left(\frac{7}{2x}\right)2x \\ 2x\left(\frac{2}{x}\right) - 2x\left(\frac{3}{2}\right) &= \left(\frac{7}{2x}\right)2x && \text{Distribute } 2x. \\ 2(2) - 3x &= 7 && \text{Denominators cancel out.} \\ 4 - 3x &= 7 \\ -3x &= 3 \\ x &= -1 \\ \text{or } \{-1\} \end{aligned}$$

The proposed solution is -1 , which is not an excluded value, so the solution set contains one number, -1 , or $\{-1\}$ written in set notation.

Try It #3

Solve the rational equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$.

Example 5 Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation: $\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$.

Solution First find the common denominator. The three denominators in factored form are x , $10 = 2 \cdot 5$, and $4x = 2 \cdot 2 \cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x = 0$ is an excluded value. Multiply the whole equation by $20x$.

$$20x\left(\frac{1}{x}\right) = \left(\frac{1}{10} - \frac{3}{4x}\right)20x$$

$$20 = 2x - 15$$

$$35 = 2x$$

$$\frac{35}{2} = x$$

The solution is $\frac{35}{2}$.

Try It #4

Solve the rational equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$.

Example 6 Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

a. $\frac{3}{x-6} = \frac{5}{x}$

b. $\frac{x}{x-3} = \frac{5}{x-3} - \frac{1}{2}$

c. $\frac{x}{x-2} = \frac{5}{x-2} - \frac{1}{2}$

Solution

- a. The denominators x and $x - 6$ have nothing in common. Therefore, the LCD is the product $x(x - 6)$. However, for this problem, we can cross-multiply.

$$\frac{3}{x-6} = \frac{5}{x}$$

$$3x = 5(x - 6) \quad \text{Distribute.}$$

$$3x = 5x - 30$$

$$-2x = -30$$

$$x = 15$$

The solution is 15. The excluded values are 6 and 0.

- b. The LCD is $2(x - 3)$. Multiply both sides of the equation by $2(x - 3)$.

$$2(x-3)\left(\frac{x}{x-3}\right) = \left(\frac{5}{x-3} - \frac{1}{2}\right)2(x-3)$$

$$\frac{2(\cancel{x-3})x}{\cancel{x-3}} = \frac{2(\cancel{x-3})5}{\cancel{x-3}} - \frac{2(x-3)}{2}$$

$$2x = 10 - (x - 3)$$

$$2x = 10 - x + 3$$

$$2x = 13 - x$$

$$3x = 13$$

$$x = \frac{13}{3}$$

The solution is $\frac{13}{3}$. The excluded value is 3.

- c. The least common denominator is $2(x - 2)$. Multiply both sides of the equation by $x(x - 2)$.

$$2(x - 2)\left(\frac{x}{x - 2}\right) = \left(\frac{5}{x - 2} - \frac{1}{2}\right)2(x - 2)$$

$$2x = 10 - (x - 2)$$

$$2x = 12 - x$$

$$3x = 12$$

$$x = 4$$

The solution is 4. The excluded value is 2.

Try It #5

Solve $\frac{-3}{2x + 1} = \frac{4}{3x + 1}$. State the excluded values.

Example 7 Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x + 1} - \frac{1}{x - 1} = \frac{2x}{x^2 - 1}$. State the excluded values.

Solution We must factor the denominator $x^2 - 1$. We recognize this as the difference of squares, and factor it as $(x - 1)(x + 1)$. Thus, the LCD that contains each denominator is $(x - 1)(x + 1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$(x - 1)(x + 1)\left(\frac{2}{x + 1} - \frac{1}{x - 1}\right) = \left(\frac{2x}{(x - 1)(x + 1)}\right)(x - 1)(x + 1)$$

$$2(x - 1) - 1(x + 1) = 2x$$

$$2x - 2 - x - 1 = 2x \quad \text{Distribute the negative sign.}$$

$$-3 - x = 0$$

$$-3 = x$$

The solution is -3 . The excluded values are 1 and -1 .

Try It #6

Solve the rational equation: $\frac{2}{x - 2} + \frac{1}{x + 1} = \frac{1}{x^2 - x - 2}$.

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as $y = mx + b$, where $m = \text{slope}$ and $b = y\text{-intercept}$. Let us begin with the slope.

The Slope of a Line

The **slope** of a line refers to the ratio of the vertical change in y over the horizontal change in x between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in **Figure 2**. The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

LEARNING OBJECTIVES

In this section you will:

- Set up a linear equation to solve a real-world application.

2.3 MODELS AND APPLICATIONS



Figure 1 Credit: Kevin Dooley

Josh is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Josh to end the course with an A? A simple linear equation will give Josh his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as \$0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges \$0.10/mi plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost C .

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. **Table 1** lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by a	$x, x + a$
Twice a number	$2x$
One number is a more than another number	$x, x + a$
One number is a less than twice another number	$x, 2x - a$
The product of a number and a , decreased by b	$ax - b$
The quotient of a number and the number plus a is three times the number	$\frac{x}{x + a} = 3x$
The product of three times a number and the number decreased by b is c	$3x(x - b) = c$

Table 1